FINITE MATRICES AND THEIR CHARACTERISTIC ROOTS

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PRELIMINARY INTRODUCTION

This thesis consists of the following ten papers:

Papers i, ii, iii, iv, v, and vii have been accepted for publication. Papers i, v and vii have been published and their reprints have been attached in this thesis.

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NOTATIONS

1. Capital Latin Letters A, B, ... denote matrices and a_{rs}, b_{rs}, ... denote, respectively, their elements in the (r,s)-th position.
2. A' denotes the transpose of A.
3. $\bar{A}$ denotes the complex conjugate of A; i.e., the element in the (r,s)-th place of $\bar{A} = \bar{a}_{rs}$.
4. $\bar{A}'$ (or A^*) denotes the conjugate transpose of A.
5. I and 0 denote, respectively, the identity and null matrices their orders to be clear from the context.
6. diag.$(\alpha_1, \alpha_2, \dots, \alpha_n)$ denotes the n-square matrix with $\alpha_1, \alpha_2, \dots, \alpha_n$ in the main diagonal and zeros elese-where.
7. $c(A)$ denotes an arbitrary characteristic root of A and $\bar{c}(A)$, its complex conjugate.
8. $c_{max}(A)$ denotes the greatest characteristic root of A.
9. $c_{min}(A)$ donotes the smallest characteristic root of A.
10. (x,y) denotes the inner product of the vectors x and y. If $x = (x_i)$, $y = (y_i)$, then $(x,y) = \sum_{i=1}^{n} x_i y_i$.
11. $\dot{+}$ denotes the direct sum.
12. The Greek letters Σ and π denote the addition and multiplication, respectively.

INTRODUCTION

Let $A = (a_{rs})$ be a square matrix of order n whose elements belong to the field of real or complex numbers. If I_n is the identity matrix of order n and λ is a scalar indeterminate, the equation obtained by equating the determinant $|A - \lambda I_n|$ to zero is called the <u>characteristic equation</u> of A. The roots, λ_i, of this equation are called the <u>characteristic roots</u> (<u>characteristic values</u>, <u>eigenvalues</u>, or <u>latent roots</u>) of A. (An arbitrary characteristic root of A to be denoted throughout by c(A)). The characteristic equation is one of the most important subjects of study and has received a great deal of attention during the last sixty or seventy years.

If A is a matrix of some special type, certain definite statements can be made about the nature of its characteristic roots. For example,

(i). if A is Hermitian (or symmetric), c(A)'s are all real;

(ii). if A is skew-Hermitian (or skew-symmetric), c(A)'s are purely imaginary or zero;

(iii). if A is unitary, c(A)'s are all in absolute value one;

(iv). if A is nilpotent, i.e., for some positive integer r, $A^r = 0$, the null matrix, c(A)'s are equal to zero;

(v). if A is idempotent, i.e., for some positive integer r, $A^r = A$, all c(A)'s are zero and/or roots of unity; (vi). if A is involutory, i.e., $A^2 = I$, c(A)'s are 1 and -1.

However, in spite of these results concerning the special types of matrices A, nothing was known about the nature of the characteristic roots of a general matrix until 1902, when I. Bendixon first attacked the problem. Since then several authors have given upper and lower limits to the characteristic roots of A. Limits for c(A), and for the real and imaginary parts of c(A) have been found by some authors in terms of the elements of A, $(A+\bar{A}')/2$ and $(A-\bar{A}')/2i$, and by others in terms of the characteristic roots of the auxiliary matrices $(A+\bar{A}')/2$, $(A-\bar{A}')/2i$ and $A\bar{A}'$.

Bendixon [2] considered a real matrix A and obtained the upper and lower limits for the real and imaginary parts of c(A). He proved the following theorem:

<u>Bendixon's Theorem</u>. If $\alpha+i\beta$ is a characteristic root of an arbitrary n-square real matrix $A = (a_{rs})$, then

$$|\beta| \leq g''\sqrt{[n(n-1)/2]} \quad , \tag{1}$$

and
$$\rho_1 \leq \alpha \leq \rho_n \quad , \tag{2}$$

where g'' is the greatest of the quantities $(|a_{rs} - a_{sr}|)/2$ and $\rho_1 \leq \rho_2 \leq \dots \leq \rho_n$ are the characteristic roots (all real) of the real symmetric matrix $(A+A')/2$.

The same year, A. Hirsch [9] extended these results to the case of a matrix A with complex elements and proved the following theorem:

Hirsch's Theorem. (a) If $\alpha+i\beta$ is a characteristic root of an n-square complex matrix $A = (a_{rs})$, and if

$$g = \max_{1\le r,s\le n} |a_{rs}|, \qquad g' = \max_{1\le r,s\le n} (|a_{rs} + \bar{a}_{sr}|)/2,$$

$$g'' = \max_{1\le r,s\le n} (|a_{rs} - \bar{a}_{sr}|)/2,$$

then

$$|\alpha+i\beta| \le ng, \qquad |\alpha| \le ng', \qquad |\beta| \le ng''. \tag{3}$$

(b) In case a_{rs} are such that the $a_{rs} + a_{sr}$ are all real, the last inequality can be replaced by

$$|\beta| \le g'' \sqrt{[n(n-1)/2]} . \tag{4}$$

(c) If $\rho_1 \le \rho_2 \le \dots \le \rho_n$ are the characteristic roots of $(A+\bar{A}')/2$, then

$$\rho_1 \le \alpha \le \rho_n . \tag{5}$$

In 1906, T.J.I'a Bromwich [4] considered the real and complex matrices. He gave a proof of Hirsch's Theorem (c) and further obtained limits for the imaginary parts of $c(A)$ in terms of the characteristic roots of $(A-\bar{A}')/2i$. He proved the following theorem:

Bromwich's Theorem. If $\alpha+i\beta$ is a characteristic root of a matrix A, and if $\mu_1 \le \mu_2 \le \dots \le \mu_n$ are the characteristic roots (necessarily real) of $(A-\bar{A}')/2i$, then

$$\mu_1 \le \beta \le \mu_n . \tag{6}$$

It was further shown by Bromwich that if A is real and if the nonzero characteristic roots of $(A-A')/2$ are of the form $\pm i\mu_1, \pm i\mu_2, \dots, \pm i\mu_t$, $(2t \le n)$, then for any characteristic root $\alpha + i\beta$ of A, the following inequality holds:

$$|\beta| \leq \max_{(r)} |\mu_r|. \tag{7}$$

It may be observed that the bounds for the real and imaginary parts of c(A) given by (5) and (6) due, respectively, to Hirsch and Bromwich are narrower than those given by (3) due to Hirsch. For, by applying the first inequality in (3), respectively, to the matrices $(A+\bar{A}')/2$ and $(A-\bar{A}')/2i$ we obtain

$$|\rho_1| \leq ng', \; |\rho_n| \leq ng', \quad |\mu_1| \leq ng'', \quad |\mu_n| \leq ng''.$$

Hence

$$-ng' \leq \rho_1 \leq \alpha \leq \rho_n \leq ng',$$

and $$-ng'' \leq \mu_1 \leq \beta \leq \mu_n \leq ng''.$$

The limit for the imaginary part of an arbitrary characteristic root of a real matrix A as given by Bromwich is also sharper than that given by Bendixon. However, the limit given by the former is in terms of the characteristic roots of $(A-A')/2i$ while that given by the latter is in terms of the elements of the matrix itself.

Again in 1909, I. Schur [14] established the inequalities (1) and (3) in a different manner, and proved the sharper result:

$$\sum_{i=1}^{n} |\lambda_i|^2 \leq \sum_{1\leq r,s\leq n} |a_{rs}|^2, \tag{8}$$

where λ_i is a characteristic root of a complex matrix A.

Hirsch's inequality (3) was improved by E.T.Browne [6] in 1930, who showed that

$$| c(A) | \leq (R + T)/2 \quad , \tag{9}$$

where $R = \max_{(r)} R_r = \max_{(r)} \sum_{s=1}^{n} |a_{rs}|$, $T = \max_{(s)} T_s = \max_{(s)} \sum_{r=1}^{n} |a_{rs}|$.

In 1937, W.V.Parker [11] further refined the above inequality. He also gave upper limits for the real and imaginary parts of c(A) in terms of the elements of the auxiliary matrices $(A+\bar{A}')/2$ and $(A-\bar{A}')/2i$, respectively. His theorem is as follows:

<u>Parker's Theorem</u> I. If $A = (a_{rs})$ is any matrix, and if

$$2S_r = \sum_{s=1}^{n} (|a_{rs}| + |a_{sr}|), \quad 2S_r' = \sum_{s=1}^{n} |a_{rs} + \bar{a}_{sr}|,$$

$$2S_r'' = \sum_{s=1}^{n} |a_{rs} - \bar{a}_{sr}|,$$

and if S, S', S'' are the greatest of the S_r, S_r', S_r''; respectively, then for any characteristic root $\lambda = \alpha + i\beta$ of A, we have

$$|\lambda| \leq S \quad \left(\leq \frac{R+T}{2}\right) \tag{10}$$

$$|\alpha| \leq S' \tag{11}$$

and

$$|\beta| \leq S''. \tag{12}$$

Another refinement to inequality (9) due to Browne was given by A.B.Farnell [7] in 1944, who showed that

$$| c(A) | \leq \sqrt{(RT)} \leq (R + T)/2 \,. \tag{13}$$

Moreover, Farnell also proved that

$$| c(A) | \leq \left(\sum_{r=1}^{n} (U_r V_r)^{1/2} \right)^{1/2} , \tag{14}$$

where $U_r = \sum_{s=1}^{n} | a_{rs} |^2$, and $V_s = \sum_{r=1}^{n} | a_{rs} |^2$.

In 1945, E.W.Barankin [1] further sharpened Farnell's result (13) by showing that

$$|c(A)| \leq \max_{(r)} (R_r T_r)^{1/2} \leq (RT)^{1/2}. \qquad (15)$$

In 1943, Parker [12] proved the following theorem:

Parker's Theorem II. If μ is a characteristic root of the matrix A and R is the greatest sum obtained for the absolute values of the elements of a row and T is the greatest sum obtained for the absolute values of the elements of a column, then

$$|\mu| \leq \min (R,T). \qquad (16)$$

This result was subsequently given by Barankin [1] and later again by A.Brauer [3] in 1946. This result had already been proved by G.Frobenius [8] in 1908 for a matrix A whose elements are real and positive. It may be observed here that inequality (16) is better than the inequality (13) due to Farnell.

During the last ten years A.Brauer [3] has proved a number of important results on the characteristic roots of a matrix A and has also discussed in details the location of these roots on the z-plane. I mention here three of his theorems:

Brauer's Theorem I. Let $A = (a_{rs})$ be an arbitrary matrix and

$$P_r = \sum_{\substack{s=1 \\ s \neq r}}^{n} |a_{rs}|, \quad Q_s = \sum_{\substack{r=1 \\ r \neq s}}^{n} |a_{rs}|.$$

Each characteristic root μ of A lies in at least one of and at least one of the n(n-1)/2 inequalities

$$|f_r(\mu) - a_{rr}^{(f_r)}| \cdot |f_s(\mu) - a_{ss}^{(f_s)}| \leq P_r^{(f_r)} P_s^{(f_s)} ,$$

$$(r,s=1,2,\ldots,n;\ r \neq s).$$

The theorems proved in these papers are also applied to stochastic matrices and to the numerical computation of the error in the approximate solutions of linear equations.

The results mentioned so far fall into two classes; most of them give inequalities for the charcteristic roots of a matrix in terms of the elements of A, $(A+\bar{A}')/2$ and $(A-\bar{A}')/2i$; the others give the inequalities in terms of the characteristic roots of $(A+\bar{A}')/2$ and $(A-\bar{A}')/2i$. But in 1928, E.T. Browne [6] gave the limits to the characteristic roots of A in terms of those of the auxiliary matrix $A\bar{A}'$. His result is as follows:

Browne's Theorem. If μ is a charcteristic root of a matrix A, and if s and G are, respectively, the smallest and the greatest charcteristic roots of $A\bar{A}'$, then

$$s \leq |\mu|^2 \leq G. \tag{17}$$

This result was extended by S.N.Roy [13] in 1954 to the case of the characteristic roots of the product matrix AB of two n-square complex matrices A and B. His result is as follows:

Roy's Theorem. If A and B are two n×n matrices one of which is nonsingular, then for all the characteristic roots c(AB), we have

$$c_{min}(A\bar{A}')c_{min}(B\bar{B}') \leq |c(AB)|^2 \leq c_{max}(A\bar{A}')c_{max}(B\bar{B}'), \tag{18}$$

where c_{min} and c_{max} stand respectively for the smallest and the largest characteristic roots (each, of course, nonnegative).

It was shown recently by Bela Sz.-Nagy [10] that the above result is valid also when both A and B are singular.

In a series of papers, I have found

(i). the limits for c(AB) in terms of the elements of A and B (Papers vi and vii);

(ii). the limits for the real and imaginary parts of c(AB) in terms of the elements of the auxiliary matrices $(A+\bar{A}')/2$, $(A-\bar{A}')/2i$, $(B+\bar{B}')/2$ and $(B-\bar{B}')/2i$, (Paper viii); and

(iii). the limits for the real and imaginary parts of c(AB) in terms of the characteristic roots of the four associated Hermitian matrices $(A+\bar{A}')/2$, $(A-\bar{A}')/2i$, $(B+\bar{B}')/2$ and $(B-\bar{B}')/2i$, (Papers ix and x).

The results thus obtained are such that if we put B = I in them we get the corresponding results for c(A), cited above.

To begin with, I first extended the result of S.N.Roy [13] to the characteristic roots of the product of two matric polynomials f(A) and g(B). This has been done in Paper v. The arguments used in this paper run parallel to those of Roy.

In Paper vi the following two inequalities have been established:

$$|c(AB)| \leq \left[\max|c(\tfrac{A+\bar{A}'}{2})| + \max|c(\tfrac{A-\bar{A}'}{2i})|\right].$$

$$\left[\max|c(\frac{B+\bar{B}'}{2})| + \max|c(\frac{B-\bar{B}'}{2i})| \right] \tag{19}$$

and

$$|c(AB)| \leq \frac{R(A) + T(A)}{2} \cdot \frac{R(B) + T(B)}{2}, \tag{20}$$

where $R(A) = \max \sum_{s=1}^{n} |a_{rs}|$, $T(A) = \max \sum_{r=1}^{n} |a_{rs}|$.

Here (19) gives an upper limit for $|c(AB)|$ in terms of the characteristic roots of $(A+\bar{A}')/2$, $(A-\bar{A}')/2i$, $(B+\bar{B}')/2$ and $(B-\bar{B}')/2i$, while (20) gives an upper limit for $|c(AB)|$ in terms of the elements of A and B. Putting $B = I$ in (20), we obtain (9).

In Paper vii, inequality (20) has been sharpened. It has been proved that

$$|c(AB)| \leq (R(A)T(A)R(B)T(B))^{1/2}. \tag{21}$$

But the method followed in establishing the above result is different from that followed in establishing (20).

Another result proved in this paper is

$$|c(AB)| \leq \min (R(A)R(B), T(A)T(B)). \tag{22}$$

In particular, if A and B are Hermitian or skew-Hermitian we have

$$|c(AB)| \leq R(A)R(B). \tag{23}$$

It may be observed that the inequalities (21) and (22) contain, respectively, the inequalities (15) and (16) as particular cases.

In Paper viii, upper limits for the real and imaginary parts of c(AB) have been found in terms of the elements of

$(A+\bar{A}')/2$, $(A-\bar{A}')/2i$, $(B+\bar{B}')/2$, $(B-\bar{B}')/2i$. It has been proved that

$$\left|\frac{c(AB)+\bar{c}(AB)}{2}\right| \leq S'(A)S'(B) + S''(A)S''(B), \qquad (24)$$

and $$\left|\frac{c(AB)-\bar{c}(AB)}{2i}\right| \leq S'(A)S''(B) + S''(A)S'(B), \qquad (25)$$

where $S_r'(A)$, $S_r''(A)$, $S_r'(B)$, $S_r''(B)$ are the sums of the absolute values of the elements in the r-th row of $(A+\bar{A}')/2$, $(A-\bar{A}')/2i$, $(B+\bar{B}')/2$, $(B-\bar{B}')/2i$, respectively, and $S'(A)$, $S''(A)$, $S'(B)$, $S''(B)$ are, respectively, the greatest of the $S_r'(A)$, $S_r''(A)$, $S_r'(B)$, $S_r''(B)$.

These results contain inequalities (11) and (12), due to Parker, as special cases.

In Paper ix, the upper and lower limits for the real and imaginary parts of c(AB), A and B being complex matrices, have been found in terms of the characteristic roots of $(A+\bar{A}')/2$, $(A-\bar{A}')/2i$, $(B+\bar{B}')/2$, and $(B-\bar{B}')/2i$. The results established here are generalisations of the inequalities (5) and (6) due to Hirsch and Bromwich, respectively. I have tried to give a detailed account , by considering separately the differeent cases when some or all the four Hermitian matrices $(A+\bar{A}')/2$, $(A-\bar{A}')/2i$, $(B+\bar{B}')/2$, and $(B-\bar{B}')/2i$ are at least positive semi-definite, the remaining being at least positive semi-definite, at least negative semi-definite or indefinite.

In Paper x, I have considered matrices A and B with real elements. Necessity for this arises only due to the fact that the nonzero characteristic roots of the skew-symmetric matrix $(A-A')/2$ must be of the form $\pm i\mu_1, \pm i\mu_2, \ldots, \pm i\mu_s$, $2s \leq n$, which means that

$$c_{max}\left(\frac{A-A'}{2i}\right) = -c_{min}\left(\frac{A-A'}{2i}\right).$$

But, for skew-Hermitian matrix $(A-\bar{A}')/2$ the nonzero characteristic roots need not be of the above form. Of course, they are purely imaginary or zero. If we put $B = I$ in the results proved in this paper we get the inequalities (2) and (7) due, respectively, to Bendixon and Bromwich.

The four papers in the beginning of this thesis, viz., Papers i, ii, iii and iv, deal with some special matrices. In Paper ii, I have proved the following results in Theory of Numbers:

<u>Theorem</u> 1. $2^{k-1} + 1 \not\equiv 0 \pmod{k}$, for any integral value of $k > 1$.

<u>Theorem</u> 2. $2^{p_1} + 2^{p_2} \not\equiv 0 \pmod{p_1 p_2}$, where p_1 and p_2 are primes except when $p_1 p_2 = 4$ or 6.

Some results on the congruences of involutory matrices have also been established there. The following is one of them:

Let A be an involutory matrix whose elements are integers (positive, negative or zero). If p is a prime or a pseudo-prime, then the two congruences

$$2(I \pm A)^p \mp 2A \equiv 2I \pmod{p}$$

hold and conversely.

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SOME NORM INEQUALITIES FOR SQUARE MATRICES

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SOME NORM INEQUALITIES FOR SQUARE MATRICES

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1. *Introduction and notation.* Let $A = (a_{ij})$ be a square matrix of order n whose elements are ordinary complex numbers. By A^* we denote the conjugate transpose of A; and by $tr\ A$, the trace, $\sum_{i=1}^{n} a_{ii}$, of A. The Frobenius norm of a matrix A, denoted by $N(A)$, is defined by

$$(1) \qquad N(A) = (tr\ AA^*)^{1/2} = \Big(\sum_{i,j=1}^{n} a_{ij}\,\bar{a}_{ij} \Big)^{1/2} = \Big(\sum_{i,j=1}^{n} |a_{ij}|^2 \Big)^{1/2}.$$

The principal properties of $N(A)$ are as follows (see Wedderburn [6]; see also Hardy-Littlewood-Polya [3;36]) :—

$$(2) \qquad N(A+B) \leqslant N(A) + N(B)$$

$$(3) \qquad N(\lambda I) = n^{1/2}\,|\lambda|,$$

$$(4) \qquad N(\lambda A) = |\lambda|\, N(A), \text{ and}$$

$$(5) \qquad N(AB) \leqslant N(A).N(B),$$

where λ is a complex number and A, B are two square matrices of order n.

R. Bellman [1] proved the following inequality employing Hölder's inequality and an identity involving multiple integrals :—

$$(6) \qquad |\alpha A + (1-\alpha) B| \geqslant |A|^{\alpha}\ |B|^{1-\alpha}$$

for any positive definite hermitian matrices A and B and $0 < \alpha < 1$, where $|A|$ denotes the determinant of the matrix A. In a recent paper L. Mirsky [4] has given an alternative proof of (6). The purpose of this paper is to establish similar inequalities with

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conditions on A and B. But, instead of the inequalities in determinants, we shall prove the results in terms of the norm of matrices.

2. *Theorem 1. For any square matrix A of order n, and any number p such that* $0 < p < I$,

(7) $\qquad n^{1/2} N[pA+(1-p)I] \geqslant p\,|tr\, A|.$

Equality holds if and only if $A=I$, the identity matrix, and $p=1$.

Proof. By a well-known theorem of I. Schur [5], there exists a unitary matrix U which reduces A to a triangular matrix T. The principal diagonal of T consists of $\lambda_1, \lambda_2, \ldots, \lambda_n$, the characteristic roots of A, not necessarily all distinct, arranged in any desired order. Then

$$UAU^* = T, \text{ and } UU^* = U^*U = I$$

imply

$$\begin{aligned}
N^2[pA+(1-p)I] &= tr\,[\{pA+(1-p)I\}\{pA+(1-p)I\}^*] \\
&= tr\,[\{pA+(1-p)I\}\{pA^*+(1-p)I\}] \\
&= tr\,[U\{pA+(1-p)I\}U^*U\{pA^*+(1-p)I\}U^*] \\
&= tr\,[\{pUAU^*+(1-p)I\}\{pUA^*U^*+(1-p)I\}] \\
&= tr\,[\{pT+(1-p)I\}\{pT^*+(1-p)I\}] \\
&\geqslant \sum_{r=1}^{n} |p\lambda_r+1-p|^2 \\
&\geqslant \sum_{r=1}^{n} |p\lambda_r|^2 \\
&= p^2 \sum_{r=1}^{n} |\lambda_r|^2 \\
&\geqslant (p^2 n/n^2)\left[\sum_{r=1}^{n} |\lambda_r|\right]^2 \\
&\geqslant (p^2/n) \left|\sum_{r=1}^{n} \lambda_r\right|^2 \\
&= (p^2/n)\,|tr\, A|^2
\end{aligned}$$

Hence $\qquad n^{1/2} N[pA + (1-p)I] \geqslant p\,|tr\, A|.$

If $A=I$ and $p=1$, both the sides of (7) are equal to n.

3. In this section we shall suppose that the given matix A is at least positive semi-definite, whose characteristic roots (necessarily real and nonnegative) are $\lambda_1, \lambda_2, \ldots, \lambda_n$, not necessarily all distinct. Here we shall prove

Theorem 2. Let the matrix A of order $n \times n$ be at least positive semi-definite hermitian. Then

(8) $\quad N[pA+(1-p)I] \geqslant pN(A), \quad 0 \leqslant p \leqslant 1,$

(9) $\quad N[pA+(1-p)I] \leqslant N(A^p), \; p=0, 1, 2, 3, \ldots$

Equality holds in both the cases when $p=1$. In (9) equality also holds when $A=I$, or $p=0$.

Proof. Since A is at least positive semi-definite hermitian matrix, there exists a unitary matrix U, such that

(10) $$U^{-1}AU=D,$$

where $D \equiv \operatorname{diag}(\lambda_2, \lambda_1, \ldots, \lambda_n)$, λ_i's are the real nonnegative characteristic roots of A.

From (10) we have

$$U^{-1}A^pU=D^p, \; p=1, 2, 3, \ldots.$$

Moreover,

$$tr\, U^{-1}AU=tr\, D=\lambda_1+\lambda_2+\ldots+\lambda_n=tr\, A.$$

Thus we have

$$\begin{aligned} N^2[pA+(1-p)I] &= tr\,[\{pA+(1-p)I\}\{pA+(1-p)I\}^*] \\ &= tr\,[\{pA+(1-p)I\}\{pA^*+(1-p)I\}] \\ &= tr\,[U^{-1}\{pA+(1-p)I\}UU^{-1}\{pA^*+(1-p)I\}U] \\ &= tr\,[\{pU^{-1}AU+(1-p)I\}\{pU^{-1}AU+(1-p)I\}] \\ &= tr\,[\{pD+(1-p)I\}\{pD+(1-p)I\}] \\ &= \sum_{r=1}^{n}(p\lambda_r+1-p)^2. \end{aligned}$$

We shall now consider two cases separately :

Case (*i*). Let $0 \leqslant p \leqslant 1$.

Since, $(p\lambda_i+1-p)^2 \geqslant p^2 \lambda_i^2$,

$$\sum_{r=1}^{n} (p\lambda_r+1-p)^2 \geqslant p^2 \sum_{r=1}^{n} \lambda_r^2$$

$$= p^2 \, tr \, (D^2)$$

$$= p^2 \, tr \, (U^{-1} A^2 U)$$

$$= p^2 \, tr \, (A^2)$$

$$= p^2 \, tr \, (AA^*) = p^2 N^2 (A).$$

Therefore, $N \, [pA + (1-p) \, I] \geqslant pN \, (A)$.

When $p=1$, both the sides reduce to $N \, (A)$, while for $p=0$ (8) reduces to $N \, (A) \geqslant 0$, which is true.

Case (*ii*). Let p be any integer greater than or equal to 1.

Since every $\lambda_i \geqslant 0$, we can employ a well-known inequality [3; 40]

$$px+1-p \leqslant x^p, \; x \geqslant 0, \; p \geqslant 1,$$

and get $p\lambda_i + 1-p \leqslant \lambda_i^p$, for $i=1, 2, \ldots, n$.

That is $\sum_{r=1}^{n} (p\lambda_r+1-p)^2 \leqslant \sum_{r=1}^{n} \lambda_r^{2p}$

$$= tr \, (D^{2p})$$

$$= tr \, (U^{-1} A^{2p} U)$$

$$= tr \, (A^{2p})$$

$$= tr \, \{(A^p) \, (A^p)^*\} = N^2 \, (A^p),$$

whence we have

$$N \, [pA+(1-p) \, I] \leqslant N \, (A^p), \; p=0, 1, 2, 3, \ldots$$

Equality holds when (*i*) $p=0$, 1, or when (*ii*) $A=I$.

4. Now we shall consider two matrices A and B, one of which is positive definite while the other is at least positive semi-

definite hermitian matrix of order n; and we shall prove the following theorem:—

Theorem 3. Let A and B be two hermitian matrices of the same order, out of which A is at least positive semi-definite, B is positive definite and p is any integer >1, then

$$(11) \qquad N[pA+(1-p)B] \leqslant cN^p(A),$$

where $c>0$ is a constant depending only on A and B.

Proof. The two hermitian matrices A and B of the same order, of which B is positive definite, can be reduced simultaneously. That is, there exists a non-singular matrix H, [2; 147], such that

$$H^*AH=D\equiv\text{diag}(\lambda_1, \lambda_2, \ldots\ldots, \lambda_n),\ H^*BH=I,$$

where $\lambda_1, \lambda_2, \ldots\ldots, \lambda_n$ are the (necessarily real) roots of the equation $|A-\lambda B|=0$. Moreover, since H^*AH is at least positive semi-definite, $\lambda_1, \lambda_2, \ldots\ldots, \lambda_n \geqslant 0$.

We now have

$$\begin{aligned} pA+(1-p)B &= p(H^*)^{-1}DH^{-1}+(1-p)(H^*)^{-1}H^{-1} \\ &= (H^*)^{-1}\{pD+(1-p)I\}H^{-1} \end{aligned}$$

so that,
$$\begin{aligned} N[pA+(1-p)B] &= N[(H^*)^{-1}\{pD+(1-p)I\}H^{-1}] \\ &\leqslant N(H^*)^{-1}N\{pD+(1-p)I\}N(H^{-1}) \\ &= N(H^*)^{-1}N(H^{-1})\left\{\sum_{r=1}^{n}(p\lambda_r+1-p)^2\right\}^{\frac{1}{2}} \\ &< N(H^*)^{-1}N(H^{-1})\left\{\sum_{r=1}^{n}\lambda_r^{2p}\right\}^{\frac{1}{2}} \\ &= N(H^*)^{-1}N(H^{-1})N(D^p) \\ &= N(H^*)^{-1}N(H^{-1})N[(H^*AH)^p] \\ &\leqslant N(H^*)^{-1}N(H^{-1})N^p(H^*)N^p(H)N^p(A) \\ &= cN^p(A), \end{aligned}$$

where $c=N(H^*)^{-1}N(H)N^p(H^*)N^p(H)$, is a positive constant depending upon A and B only.

In conclusion, I wish to express my indebtedness to Dr. S. M. Shah for his helpful suggestions and valuable criticism.

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ON INVOLUTORY MATRICES

(Accepted for publication in the American Mathematical Monthly)

1. Introduction.

Let I denote an identity matrix and 0, a null matrix whose orders will be clear from the context. A square matrix $A = (a_{ij})$ is called involutory if $A^2 = I$. Throughout this paper the elements of A will belong to the field F of real numbers.

If X and Y be two matrices of the same order and that all the elements of $X-Y$ be integer multiples of p, where p is an integer greater than 1, then we shall write $X \equiv Y \pmod p$. Following Lehmer we shall call an integer p, a pseudoprime, (for results on pseudoprimes see [2] and references given therein), if $2^p \equiv 2 \pmod p$ and p is not a prime.

In section 2 of this paper two theorems on the Theory of Numbers have been proved, one of which will be used to prove Theorems 9 and 10 of section 4. In section 3 some relationships between involutory and idempotent matrices have been established.

2. We first prove the following two theorems on Theory of Numbers:

Theorem 1. $2^{k-1} + 1 \not\equiv 0 \pmod k$, for any integral value of $k > 1$.

To prove this theorem, let us consider the following two cases:

<u>Case (i).</u> <u>Let k be a positive even integer.</u>

Suppose there exists an integer y such that

$$2^{k-1} + 1 = ky. \tag{1}$$

For all positive even integers k, the left-hand side of (1) is always an odd integer. But the right-hand side is an even integer. Hence our supposition is wrong and $2^{k-1} + 1 \not\equiv 0 \pmod k$, where k is a positive even integer.

<u>Case (ii).</u> <u>Let k be an odd integer greater than 1.</u>

Suppose $k = p_1^{a_1} p_2^{a_2} \cdots p_r^{a_r}$, where p_r's are all distinct primes different from 2.

Now $p_i - 1 = 2^{m_i} t_i$, t_i being odd, for $i = 1, 2, \ldots, r$.

Without loss of generality we can suppose that $m_1 = \min(m_1, m_2, \ldots, m_r)$.

Therefore, since $p_i \equiv 1 \pmod{2^{m_1}}$, for $i = 1, 2, \ldots, r$,

$$k - 1 = 2^{m_1} u, \text{ for some integer } u.$$

Now suppose that $2^{k-1} \equiv -1 \pmod k$, and therefore $2^{2^{m_1} u} \equiv -1 \pmod{p_1}$.

Raising both the sides of the above congruence to the odd power t_1, we have

$$2^{2^{m_1} u t_1} \equiv -1 \pmod{p_1},$$

or $$2^{(p_1-1)u} \equiv -1 \pmod{p_1}.$$

But, by Fermat's Theorem $2^{p_1-1} \equiv 1 \pmod{p_1}$, which gives

$$2^{(p_1-1)u} \equiv 1 \pmod{p_1}.$$

This gives a contradiction, and the result is, therefore, proved in this case also.

<u>Theorem</u> 2. $2^{p_1} + 2^{p_2} \not\equiv 0 \pmod{p_1p_2}$, <u>where</u> p_1 <u>and</u> p_2 <u>are primes, except when</u> $p_1p_2 = 4$ <u>or</u> 6.

<u>Proof</u>. To prove this theorem consider the following cases:

<u>Case</u> (i). p_1 and p_2 are odd.

Assume that $2^{p_1} + 2^{p_2} \equiv 0 \pmod{p_1p_2}$, so that

$$2^{p_1} + 2^{p_2} \equiv 0 \pmod{p_1}. \tag{2}$$

Also, by Fermat's Theorem, we have

$$2^{p_1} - 2 \equiv 0 \pmod{p_1}. \tag{3}$$

(2) and (3) give

$$2^{p_2} + 2 \equiv 0 \pmod{p_1},$$

or
$$2^{p_2} \equiv -2 \pmod{p_1},$$

or
$$(2^{p_2})^{p_1} \equiv -2^{p_1} \pmod{p_1}, \text{ since } p_1 \text{ is odd,}$$

or
$$2^{p_1p_2} + 2^{p_1} \equiv 0 \pmod{p_1}.$$

The above congruence with the help of (3) gives

$$2^{p_1p_2-1} + 1 \equiv 0 \pmod{p_1}.$$

Similarly, we have

$$2^{p_1p_2-1} + 1 \equiv 0 \pmod{p_2}, \text{ and so}$$
$$2^{p_1p_2-1} + 1 \equiv 0 \pmod{p_1p_2},$$

contrary to Theorem 1.

Case (ii). $p_1 = 2$, $p_2 > 3$.

Assuming that $2^{p_1} + 2^{p_2} \equiv 0 \pmod{p_1 p_2}$, we have $2^{p_1} + 2^{p_2} \equiv 0 \pmod{p_2}$, so that $2^{p_2} + 4 \equiv 0 \pmod{p_2}$.

But, by Fermat's Theorem

$$2^{p_2} - 2 \equiv 0 \pmod{p_2}.$$

Hence, $6 \equiv 0 \pmod{p_2}$, which contradicts $p_2 > 3$.

This completes the proof of Theorem 2.

Theorem 3. The involutory matrix A of order n is similar to $I_p \dotplus (-I_{n-p})$, where p depends on A and $\dotplus$ denotes the direct sum.

Proof. Since $A^2 = I$, it satisfies $x^2 - 1 = 0$ and the minimum polynomial of A divides x^2-1. It can be either $x-1$, $x+1$ or x^2-1. But, if A is neither the identity matrix I nor is it $-I$, then it satisfies neither $x-1=0$, nor $x+1=0$. Therefore, assuming that $A \neq I, -I$, its minimal polynomial is $x^2 - 1$.

Thus the elementary divisors of A are the monic polynomials $x-1$ and $x+1$, and they are relatively prime. The companion matrix for each factor $x-1$ of the characteristic polynomial of A is the matrix (1) and for $x+1$ it is the matrix (-1). Hence, by a well known result, [3; p. 241] , A is similar to $I_p \dotplus (-I_q)$, where $p+q = n$. This completes the proof.

By simple calculations it can be seen that the matrices $(I+A)/2$ and $(I-A)/2$ are idempotent. Therefore for any positive integer k, the matrices $\left(\frac{I+A}{2}\right)^k$ and $\left(\frac{I-A}{2}\right)^k$ are also idempotent. Moreover, since $\left(\frac{I+A}{2}\right)^k\left(\frac{I-A}{2}\right)^k = \left(\frac{I-A}{2}\right)^k\left(\frac{I+A}{2}\right)^k = 0$, $\left(\frac{I+A}{2}\right)^k$ and $\left(\frac{I-A}{2}\right)^k$

are orthogonal idempotent. Thus we have:

Theorem 4. If the matrix A is involutory, the matrices $\left(\frac{I+A}{2}\right)^k$ and $\left(\frac{I-A}{2}\right)^k$ are idempotent, k being an arbitrary positive integer.

Theorem 5. The idempotent matrices $\left(\frac{I+A}{2}\right)^k$ and $\left(\frac{I-A}{2}\right)^k$ are orthogonal.

Since the rank of the sum of two orthogonal idempotent matrices is the sum of their ranks, [1; p. [illegible]],

$$\rho\left\{\left(\frac{I+A}{2}\right)^k\right\}+\rho\left\{\left(\frac{I-A}{2}\right)^k\right\} = \rho\left\{\left(\frac{I+A}{2}\right)^k+\left(\frac{I-A}{2}\right)^k\right\},$$ ρ denotes the rank,

$$= \rho\left[2^{-k}\left\{2I + 2\binom{k}{2}A^2 + 2\binom{k}{4}A^4 + \dots\right\}\right]$$

$$= \rho\left[I\right] = n.$$

But $\rho\{(I+A)^k\} = \rho\left\{\left(\frac{I+A}{2}\right)^k\right\}$ and $\rho\{(I-A)^k\} = \rho\left\{\left(\frac{I-A}{2}\right)^k\right\}$.

Hence we have:

Theorem 6. $\rho\{(I+A)^k\}+\rho\{(I-A)^k\} = n$, the order of A, for any positive integer k.

We now establish the following theorem:

Theorem 7. For any involutory matrix A of order n, such that $\rho(I+A) = r$, the characteristic polynomial

$$|\lambda I - A| = (\lambda - 1)^r(\lambda + 1)^{n-r}.$$

Proof. From Theorem 3, A is similar to $I_p \dotplus (-I_{n-p})$. Therefore, $|\lambda I-A| = (\lambda - 1)^p(\lambda + 1)^{n-p}$.

Since $\frac{I+A}{2}$ is idempotent and $\rho\left(\frac{I+A}{2}\right) = \rho(I+A) = r$, it is similar to $I_r \dotplus 0_{n-r}$, [1; p. 38].

Therefore $\left|\lambda I - \frac{I+A}{2}\right| = (\lambda - 1)^r \lambda^{n-r}$.

But, $\left|\lambda I - \frac{I+A}{2}\right| = 2^{-n}\left|(2\lambda - 1)I-A\right|$

$$= 2^{-n}(2\lambda - 1 - 1)^{p}(2\lambda - 1 + 1)^{n-p}$$

$$= (\lambda - 1)^{p}(\lambda)^{n-p}.$$

Therefore, $p=r$, the rank of $(I+A)$.

4. We now establish some results on the congruences of involutory matrices.

<u>Theorem 3. Let A be an involutory matrix whose elements are integers (positive, negative, or zero). If p is a prime or a pseudoprime, then both congruences</u>

$$2(I \pm A)^{p} \mp 2A \equiv 2I \pmod{p},$$

<u>hold and conversely.</u>

<u>Proof.</u> If p is a prime or a pseudoprime, $2^{p} \equiv 2 \pmod{p}$, and by hypothesis the elements of A are all integers. Hence

$$2(I \pm A)^{p} \mp 2A - 2I = 2\left[I \pm \binom{p}{1}A + \binom{p}{2}A^{2} \pm \binom{p}{3}A^{3} + \ldots \right] \mp 2A - 2I$$

$$= 2\left[\left\{ \binom{p}{0} + \binom{p}{2} + \ldots \right\} I \pm \left\{ \binom{p}{1} + \binom{p}{3} + \ldots \right\} A \right] \mp 2A - 2I$$

$$= (2^{p} - 2)I \pm (2^{p} - 2)A$$

$$\equiv 0 \pmod{p}.$$

i.e., $2(I \pm A)^{p} \mp 2A \equiv 2I \pmod{p}$.

Conversely, if $2(I \pm A)^{p} \mp 2A \equiv 2I \pmod{p}$, the elements of the matrix $2(I \pm A)^{p} \mp 2A - 2I$ or $(2^{p}-2)I \pm (2^{p}-2)A$ are divisible by p. Hence $2^{p} \equiv 2 \pmod{p}$, and so either p is a prime or it must be a pseudoprime.

<u>Theorem 3.</u> <u>Let A be an involutory matrix, whose elements are integers, (positive, negative or zero). If p is a prime or a pseudoprime, then both congruences</u>

$$\left\{2(I \pm A)\right\}^p \mp 2A \equiv 2I \pmod{p},$$

<u>hold and conversely.</u>

<u>Proof.</u> By hypothesis the elements of A are all integers. Moreover, if p is a prime or a pseudoprime, then $2^p \equiv 2 \pmod{p}$, and

$$\left\{2(I \pm A)\right\}^p \mp 2A - 2I = 2^p\left[I \pm \binom{p}{1}A + \binom{p}{2}I \pm \ldots\right] \mp 2A - 2I$$

$$= 2^p\left[\left\{\binom{p}{0} + \binom{p}{2} + \ldots\right\}I \pm \left\{\binom{p}{1} + \binom{p}{3} + \ldots\right\}A\right] \mp 2A - 2I$$

$$= (2^{2p-1} - 2)I \pm (2^{2p-1} - 2)A$$

$$= (2^p - 2)(2^{p-1}+1)I \pm (2^p-2)(2^{p-1}+1)A$$

$$\equiv 0 \pmod{p}.$$

Hence $\left\{2(I \pm A)\right\}^p \mp 2A \equiv 2I \pmod{p}$.

Conversely, if $\left\{2(I \pm A)\right\}^p \mp 2A \equiv 2I \pmod{p}$, the elements of the matrix $(2^p-2)(2^{p-1}+1)(I \pm A)$ are all (integer) multiples of p. It has been proved in Theorem 1 that $2^{p-1} \not\equiv -1 \pmod{p}$. Therefore,

$$2^p - 2 \equiv 0 \pmod{p}, \text{ or } 2^p \equiv 2 \pmod{p}.$$

Hence, either p is a prime or it must be a pseudoprime.

Similarly we can prove the following theorem:

<u>Theorem 4.</u> <u>Let A be an involutory matrix whose elements are integers, (positive, negative or zero). If p is a prime or a pseudoprime, then both congruences</u>

$$2(I \pm A)^{2p-1} \mp 2A \equiv 2I \pmod p$$

<u>hold and conversely</u>.

5. To show that there exist matrices satisfying the hypotheses of Theorems 8, 9 and 10, we consider the following examples:

<u>Example</u> 1. Let p=7, n=2 and $A = \begin{bmatrix} 5 & 3 \\ -8 & -5 \end{bmatrix}$

Then $2(I+A)^7 - 2A = \begin{bmatrix} 768 & 384 \\ -1024 & -512 \end{bmatrix} - \begin{bmatrix} 10 & 6 \\ -16 & -10 \end{bmatrix}$

$= \begin{bmatrix} 758 & 378 \\ -1008 & -502 \end{bmatrix} \equiv 2I \pmod 7.$

Also $2(I-A)^7 + 2A = \begin{bmatrix} -512 & -384 \\ 1024 & 768 \end{bmatrix} + \begin{bmatrix} 10 & 6 \\ -16 & -10 \end{bmatrix}$

$= \begin{bmatrix} -502 & -378 \\ 1008 & 758 \end{bmatrix} \equiv 2I \pmod 7.$

<u>Example</u> 2. Let p=5. n=3 and $A = \begin{bmatrix} -1 & 4 & -2 \\ -4 & 9 & -4 \\ -8 & 16 & -7 \end{bmatrix}$

Then $\left\{2(I+A)\right\}^5 - 2A = \begin{bmatrix} 0 & 2048 & -1024 \\ -2048 & 5120 & -2048 \\ -4096 & 8192 & -3072 \end{bmatrix}$

$- \begin{bmatrix} -2 & 8 & -4 \\ -8 & 18 & -8 \\ -16 & 32 & -14 \end{bmatrix}$

$= \begin{bmatrix} 2 & 2040 & -1020 \\ -2040 & 5102 & -2040 \\ -4080 & 8160 & -3058 \end{bmatrix}$

$\equiv 2I \pmod 5.$

Also $2(I-A)^{2p-1} + 2A = \begin{bmatrix} 1024 & -2048 & 1024 \\ 2048 & -4096 & 2048 \\ 4096 & -8192 & 4096 \end{bmatrix}$

$$+ \begin{bmatrix} -2 & 8 & -4 \\ -8 & 18 & -8 \\ -16 & 32 & -14 \end{bmatrix}$$

$$= \begin{bmatrix} 1022 & -2040 & 1020 \\ 2040 & -4078 & 2048 \\ 4080 & -8160 & 4082 \end{bmatrix}$$

$$\equiv 2I \pmod{5}.$$

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ON INCIDENCE MATRICES.

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1. Introduction. The n, k, λ problem, viz., to arrange n elements into n sets such that (i) each set contains exactly k distinct elements and (ii) every pair of sets has exactly $\lambda = k(k-1)/(n-1)$ elements in common ($0 < \lambda < k < n$), has drawn the attention of many writers, (e.g., [4], [6] and references therein). The significance of this problem has been described in details in [3], and the possibility or impossibility of the solution of the n,k, λ problem has been discussed in details by Bruck and Ryser [2], Chowla and Ryser [3] and Shrikhande [7]. The object of this paper is to consider the nature of the characteristic roots of the incidence matrices, and to find the number of linearly independent solutions of the n,k, λ problem, if there exists one solution.

To form an incidence matrix let the elements $x_1, x_2, \ldots, x_n$ be arranged in a row and let the n sets $s_1, s_2, \ldots, s_n$ be arranged in a column. An incidence matrix A is formed by putting 1 or 0 in its (i,j)-th position according as x_j occurs in s_i or not. This matrix A is of order n×n and is composed entirely of zeros or ones. Moreover, it satisfies the matrix equation $AA' = A'A = B$ where A' denotes the transpose of A and B is a symmetric matrix of order n with k in the main diagonal and $\lambda = k(k-1)/(n-1)$ in all other positions. The matrix A is nonsingular with determinant $|A| = \pm k(k - \lambda)^{(n-1)/2}$, [6].

2. <u>Theorem 1. The characteristic roots of the incidence matrix A lie in the annular region</u>

$$\sqrt{(k - \lambda)} \leq |z| \leq k .$$

<u>Proof</u>. In order to prove the above theorem let us first evaluate the characteristic determinant $|B - xI|$ of B.

$$\begin{aligned} |B - xI| &= |(k-\lambda)I + \lambda S - xI| \\ &= |\lambda S + (k-\lambda-x)I| \\ &= \lambda^n |S + \frac{k-\lambda-x}{\lambda} I|, \end{aligned}$$

where S is the matrix of all ones. It is of rank unity and can be put as $S = R'R$, where R is the 1xn matrix (1,1,...,1). Now, applying Lemma II, proved by Roth [5] , we have :

$$|B - xI| = \lambda^n \left[|\frac{k-\lambda-x}{\lambda} I| + R(\frac{k-\lambda-x}{\lambda} I)^A R' \right] ,$$

where $(\frac{k-\lambda-x}{\lambda} I)^A$ denotes the adjointof $\frac{k-\lambda-x}{\lambda} I$.

$$\begin{aligned} &= \lambda^n \left[(\frac{k-\lambda-x}{\lambda})^n + R \left\{ (\frac{k-\lambda-x}{\lambda})^{n-1} I \right\} R' \right] \\ &= \lambda^n \left[(\frac{k-\lambda-x}{\lambda})^n + n(\frac{k-\lambda-x}{\lambda})^{n-1} \right] \\ &= \lambda^n (\frac{k-\lambda-x}{\lambda})^{n-1} \left(\frac{k-\lambda-x}{\lambda} + n \right) \end{aligned}$$

The characteristic roots of B are, then, given by

$$(\frac{k-\lambda-x}{\lambda})^{n-1} \left(\frac{k-\lambda-x}{\lambda} + n \right) = 0, \text{ since } \lambda \neq 0.$$

Therefore, if $(\frac{k-\lambda-x}{\lambda})^{n-1} = 0$, $x = k-\lambda$ with multiplicity n-1 and if $\frac{k-\lambda-x}{\lambda} + n = 0$, $x = k - \lambda(1-n) = k+k(k-1)=k^2$.

Thus the characteristic roots of B are $k-\lambda$ with multiplicity n-1 and k^2.

Now, by a well-known result due to Browne [1],

$$c_{min}(AA') \leq |c(A)|^2 \leq c_{max}(AA'),$$

where c(A) denotes a characteristic root of A, and c_{min} and c_{max} stand, respectively, for the smallest and greatest characteristic roots.

Therefore, $c_{min}(B) \leq |c(A)|^2 \leq c_{max}(B)$

or, $k - \lambda \leq |c(A)|^2 \leq k^2$.

This completes the proof of the theorem.

3. Factorizations of the incidence matrices. The matrix B is symmetric and positive definite. Therefore, by Toeplitz Factorization Theorem there exists a unique triangular matrix T with zeros above the main diagonal, the diagonal elements of T being all positive, such that

$$B = TT'$$

or, $AA' = TPP'T'$, where P is an orthogonal matrix.

Therefore, A can be factored as TP, the factor T is unique while P can be any orthogonal matrix.

Further, since B is symmetric, it is possible to find two orthogonal matrices P and Q such that

$$B = PD^2P', \text{ where } D \equiv \text{diag.}(k, \sqrt{k-\lambda}, \ldots, \sqrt{k-\lambda}),$$

$$= PDQQ'DP'$$

i.e., $$AA' = (PDQ)(PDQ)'.$$

Therefore, $$A = P.\ \mathrm{diag}(\ k,\ \sqrt{k-\lambda}\ ,\ldots,\ \sqrt{k-\lambda}\ ).Q.$$

4. <u>The solutions of the n, k, λ problem</u>. It has been shown in [6] and [4] that under certain conditions the matrix equation $AA' = B$ has a solution, such that A is an incidence matrix. In this section we shall prove

<u>Theorem</u> 2. <u>If A is an incidence matrix satisfying the matrix equation</u>

$$A'A = AA' = B \tag{1}$$

<u>then there are</u> $(n^2-n+2)/2$ <u>linearly independent solutions of (1), all of them being incidence matrices</u>.

To prove the above result let us first establish the following lemma:

<u>Lemma</u>. If E_{ij} is an elementary matrix obtained by interchanging the i-th and j-th rows of the identity matrix I, then the set of matrices E_{ij}, $(1 \leq i,j \leq n)$, is linearly independent.

Let the different elementary matrices E_{ij}, $(1 \leq i,j \leq n)$, be written in the following order:

$$I; E_{12}, E_{13}, \ldots, E_{1n};\ E_{23}, E_{24}, \ldots, E_{2n};\ \ldots\ ;\ E_{n-1,n}\ .$$

The number of the above matrices is

$$1 + (n-1) + (n-2) + \ldots + 2 + 1 = 1 + \frac{n(n-1)}{2}$$

$$= \frac{n^2-n+2}{2}\ .$$

Now suppose that there are some nonzero scalar c's, such that

$$c_{11}I + (c_{12}E_{12} + \dots + c_{1n}E_{1n}) + \dots + c_{n-1,n}E_{n-1,n}$$
$$= \; 0 \text{ ,the null matrix.} \qquad (2)$$

The left-hand side of (2) simplifies to

$$\begin{bmatrix} d_1 & c_{12} & c_{13} & \dots & c_{1n} \\ c_{12} & d_2 & c_{23} & \dots & c_{2n} \\ c_{13} & c_{23} & d_3 & \dots & c_{3n} \\ \cdot & \cdot & \cdot & \dots & \cdot \\ c_{1n} & c_{2n} & c_{3n} & \dots & d_n \end{bmatrix} ,$$

where d's , the elements in the main diagonal, are such that the sum of the elements of each row or column is equal to

$$c_{11} + (c_{12} + c_{13} + \dots + c_{1n}) + (c_{23} + c_{24} + \dots + c_{2n}) + \dots + c_{n-1,n} .$$

But, since the above matrix reduces to the null matrix, therefore, considering the elements above the main diagonal, we have

$$c_{12} = c_{13} = \dots = c_{1n} = 0;$$
$$c_{23} = c_{24} = \dots = c_{2n} = 0;$$
$$\dots\dots\dots\dots\dots\dots$$

and $\quad c_{n-1,n} = 0.$

Putting these values in the main diagonal, we have $c_{11} = 0$.

Thus equation (2) implies that c_{11} , c_{12} , ... , $c_{n-1,n}$ are all zero. Hence the $(n^2 - n + 2)/2$ matrices under consideration are linearly independent.

Now, to prove the theorem we observe that, if A is an incidence matrix satisfying (1), then $E_{ij}A$ is also a solution of (1), for $(E_{ij}A)'(E_{ij}A) = A'E_{ij}E_{ij}A = A'A = B$.

Moreover, $E_{ij}A$, the matrix A with the i-th and j-th rows interchanged, is also an incidence matrix. Thus if A is an incidence matrix satisfying (1), there are $(n^2 - n + 2)/2$ incidence matrices

$$A;\ E_{12}A,\ E_{13}A,\ \ldots,\ E_{1n}A;\ \ldots;\ E_{n-1,n}A;$$

and they form a linearly independent set of solutions of (1).

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CHARACTERISTIC ROOTS OF SEMI-MAGIC SQUARE MATRICES.

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In this paper we consider the matrices formed by the magic and semi-magic squares. A square matrix $A = (a_{rs})$ of order n is called a magic square matrix if

$$\sum_{r=1}^{n} a_{rs} = \sum_{r=1}^{n} a_{sr} = S(A), \text{ say, for } s=1,2,\dots,n; \qquad (1)$$

and $$\sum_{r=1}^{n} a_{rr} = \sum_{r=1}^{n} a_{r,n-r+1} = S(A). \qquad (2)$$

But, if only (1) holds, A is said to be semi-magic square matrix, and following L.M.Weiner, will be called an S-matrix of order n. In [4] Weiner has considered the matric algebra R_n of S-matrices of order n and has also determined the structure of the algebra R_n. The main purpose of this paper is to determine the bounds for the characteristic roots of S-matrices. Throughout this paper the entries of the S-matrices lie in the field of characteristic zero.

We first prove the following:

Theorem 1. *If A is in G', the set of all nonsingular S-matrices of order n, then A^{-1} is also in G', and* $S(A^{-1}) = [S(A)]^{-1}$.

Proof. To each element A of G', there exists a matrix $B=(b_{ij}) = A^{-1}$, such that $AB = BA = I$, the identity matrix. Considering $AB=I$, we have $\sum_{r=1}^{n} a_{ir}b_{rj} = \delta_{ij}$, δ_{ij} is the Kronecker symbol, which equals 1 when $i=j$, and equals 0 when $i \neq j$.

Thus $\sum_{i=1}^{n}\sum_{r=1}^{n} a_{ir}b_{rj} = \sum_{i=1}^{n}\delta_{ij}$,

or, $\sum_{r=1}^{n} b_{rj}\sum_{i=1}^{n} a_{ir} = 1$,

or, $S(A)\sum_{r=1}^{n} b_{rj} = 1$.

Therefore, $\sum_{r=1}^{n} b_{rj} = 1/S(A)$, a fixed quantity, for $j=1,2,\ldots,n$.

Similarly, proceeding with $BA = I$, it can be seen that

$\sum_{j=1}^{n} b_{rj} = 1/S(A)$, for $r=1,2,\ldots,n$.

Thus $B = A^{-1}$ belongs to G' and is such that $S(A^{-1}) = [S(A)]^{-1}$. This completes the proof.

It may be observed here that the set G' of all nonsingular S-matrices of the same order is a non-abelian multiplicative group. Further, it can be observed that every element A, distinct from the zero element (null matrix), belonging to G, the set of all S-matrices of the same order, generates a cyclic group of infinite order with addition as the rule of combination.

We now prove a theorem which gives an upper bound for the absolute value of any characteristic root of an S-matrix, A, with positive numbers as its elements.

<u>Theorem</u> 2. <u>If μ is any characteristic root of A, then</u>

$$|\mu| \leq S(A), \tag{3}$$

i.e., <u>the absolute value of μ is not greater than the sum of the elements of A along any row or column</u>.

<u>Proof</u>. By a well known theorem of I. Schur [3] there exists a unitary matrix $U = (u_{ij})$ which transforms A into a triangular

matrix T. The principal diagonal of T consists of the characteristic roots $\lambda_1, \lambda_2, \ldots, \lambda_n$ of A, not necessarily all distinct.

Then from

$$T = UAU^* \text{ and } UU^* = U^*U = I,$$

it follows that

$$t_{ij} = \sum_{r,s=1}^{n} u_{ir} a_{rs} \bar{u}_{js},$$

where t_{ij} is the element of T in the (i,j)-th place.

That is, the elements of T are of the form $\sum_{r,s=1}^{n} a_{rs} x_r \bar{x}_s$, [1;150],

where the set of (complex) numbers $(x_1, x_2, \ldots, x_n) \neq (0, 0, \ldots, 0)$ is such that $\sum_{i=1}^{n} x_i \bar{x}_i = 1$.

Let the absolute values of x_i be denoted by ζ_i, (i=1,2,...,n). Then the set $(\zeta_1, \zeta_2, \ldots, \zeta_n)$ is the set of real numbers such

that $\sum_{i=1}^{n} \zeta_i^2 = 1$. Also, since ζ_i, ζ_j are all real,

$$\zeta_i \zeta_j \leq \tfrac{1}{2}(\zeta_i^2 + \zeta_j^2).$$

Hence,
$$
\begin{aligned}
|t_{ii}| &= \Big| \sum_{r,s=1}^{n} a_{rs} x_r \bar{x}_s \Big| \\
&\leq \sum_{r,s=1}^{n} |a_{rs}| . |x_r| . |\bar{x}_s| \\
&= \sum_{r,s=1}^{n} a_{rs} \zeta_r \zeta_s \\
&\leq \tfrac{1}{2} \sum a_{rs} (\zeta_r^2 + \zeta_s^2) \\
&= \tfrac{1}{2} \Big[\sum_{r=1}^{n} \zeta_r^2 \sum_{s=1}^{n} a_{rs} + \sum_{s=1}^{n} \zeta_s^2 \sum_{r=1}^{n} a_{rs} \Big] \\
&= \tfrac{1}{2} \Big[S(A) \sum_{r=1}^{n} \zeta_r^2 + S(A) \sum_{s=1}^{n} \zeta_s^2 \Big] = S(A).
\end{aligned}
$$

Therefore, $|t_{ii}| \leq S(A)$.

Hence, $|\mu_i| \leq S(A)$ for i = 1,2,...,n.

This establishes the theorem.

We now generalise the above theorem in the following form:

<u>Theorem</u> 3. <u>Let</u> $r(x) = f_1(x)/f_2(x)$ <u>be a rational function of the scalar indeterminate</u> x <u>and</u> A <u>be an S-matrix, with positive numbers as its elements, and</u> $f_2(A)$ <u>is nonsingular. If</u> μ <u>is a characteristic root of</u> A<u>, then for any characteristic root</u> $r(\mu)$ <u>of</u> $r(A)$,

$$|r(\mu)| \leq r\{S(A)\} . \tag{4}$$

<u>Proof</u>. Since the S-matrices of the same order form an algebra, [Theorem 1;4] , $f_1(A)$ and $f_2(A)$ are S-matrices, the latter being nonsingular. It may, further, be verified that

$$S\{f_1(A)\} = f_1\{S(A)\} , \text{ and } \quad S\{f_2(A)\} = f_2\{S(A)\},$$

whence

$$\begin{aligned} S\{r(A)\} &= S\{f_1(A)\,(f_2(A))^{-1}\} \\ &= S\{f_1(A)\}S\{(f_2(A))^{-1}\} \\ &= S\{f_1(A)\}\{S(f_2(A))\}^{-1} \\ &= f_1\{S(A)\}/f_2\{S(A)\} \\ &= r\{S(A)\} . \end{aligned}$$

Thus $r(A) = f_1(A)/f_2(A)$ is an S-matrix. Also, by a well known theorem of Frobenius, [2;22,23] , if μ is a characteristic root of A, $r(\mu)$ is a characteristic root of $r(A)$. In order to see that $r(\mu)$ is defined, we observe that since the matrix $f_2(A)$ is nonsingular, no characteristic root of $f_2(A)$ is zero. That is, $f_2(\mu) \neq 0$, for it is a characteristic root of $f_2(A)$.

Now, applying (3), we have

$$|r(\mu)| \leq r\{S(A)\},$$

and the theorem is proved.

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A THEOREM ON THE CHARACTERISTIC ROOTS OF MATRICES

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1. In this paper I prove a theorem on the characteristic roots of matrices, which contains the results proved by Browne [1], Ishaq [2], and Roy [3] as special cases.

In this paper it shall be assumed that the elements of the matrices are real or complex numbers. Throughout the paper a capital letter, say A, will denote a square matrix of order $n > 1$, A' its transpose, A^* its conjugate transpose, a_{ij} its element in the (i, j)—th place, $\bar{a}_{ij}$ the conjugate of a_{ij}, and $c(A)$ will denote the characteristic root of A. Moreover, I and O will stand for the identity and null matrices respectively whose orders will be clear from the context.

2. Theorem. *Let $f(x)$ and $g(x)$ be two arbitrary polynomials in x with complex coefficients and let A and B be two $n \times n$ matrices such that at least one of the two matrices $f(A)$ or $g(B)$ is non-singular. Then for all the characteristic roots $c\{f(A)g(B)\}$, we have*

$$
\begin{aligned}
c_{min}[f(A)\{f(A)\}^*]\, c_{min}[g(B)\{g(B)\}^*] \\
< c[f(A)g(B)]\, \bar{c}[f(A)\, g(B)] \\
\leqq c_{max}[f(A)\{f(A)\}^*]\, c_{max}[g(B)\{g(B)\}^*],
\end{aligned}
$$

where c_{min} and c_{max} denote respectively the minimum and the maximum characteristic roots

Proof. For any square matrices A and B the matrices $f(A)\{f(A)\}^*$ and $g(B)\{g(B)\}^*$ are Hermitian and are at least positive semi-definite, all of whose characteristic roots are real and non-negative. If we further assume that $f(A)$ is non-singular, then $f(A)\{f(A)\}^*$ becomes positive definite and its characteristic roots are all real and positive.

If $D(\lambda)$ denotes a diagonal matrix with the (real) characteristic roots $\lambda_1, \lambda_2, \ldots, \lambda_n$, of $f(A)\{f(A)\}^*$ in the main diagonal and zeros elsewhere, then it is possible to find two unitary matrices P_A and Q_A such that $f(A)\{f(A)\}^* = P_A D(\lambda) P_A^* = P_A \triangle Q_A Q_A^* \triangle P_A^*$ where $\triangle$ is a diagonal matrix such that $\triangle^2 = D(\lambda)$ so that

(1) $f(A) = P_A \triangle Q_A$,

Similarly corresponding to the matrix $g(B)$ it is possible to find two unitary matrices P_B and Q_B such that

(2) $g(B) = P_B \nabla Q_B$,

where ∇ is a diagonal matrix such that $\nabla^2 = D(\mu)$ and $D(\mu)$ is the diagonal matrix with $\mu_1, \mu_2,., \mu_n$, the real and non-negative characteristic roots of the Hermitian matrix $g(B)\{g(B)\}^*$, in the main diagonal and zeros elsewhere.

Therefore, for any characteristic root of $f(A)g(B)$, we have :

$$\begin{aligned} c[f(A)g(B)] &= c\,[P_A \triangle Q_A P_B \nabla Q_B] \\ &= c\,[\triangle Q_A P_B \nabla Q_B P_A], \text{ for } c(AB) = c(BA). \\ &= c\,[\triangle R \nabla S], \end{aligned}$$

where $R = Q_A P_B$ and $S = Q_B P_A$. Moreover, R and S, being the product of unitary matrices, are themselves unitary.

Now, if σ is a characteristic root of $f(A)g(B)$, there exists a vector $(z_1, z_2, \ldots\ldots, z_n) \neq (0, 0, \ldots\ldots, 0)$ with complex coordinates such that the following linear equations are satisfied :

$$(3) \quad \sigma z_i = \sum_{j,k=1}^{n} \sqrt{\lambda_i}\, r_{ij} \sqrt{\mu_j}\, s_{jk} z_k, \quad (i = 1, 2, \ldots\ldots, n)$$

Dividing the above equations by $\sqrt{\lambda_i}$, $(i = 1, 2, \ldots n)$, which are not zero, we have :

$$(3') \quad \frac{\sigma z_i}{\sqrt{\lambda_i}} = \sum_{j,k=1}^{n} r_{ij} \sqrt{\mu_j}\, s_{jk} z_k, \quad (i = 1, 2, \ldots\ldots, n).$$

On taking the conjugate of both the sides of the equations (3′), we have :

$$(4) \quad \frac{\bar{\sigma} \bar{z}}{\sqrt{\lambda_i}} \sum_{l,m=1}^{n} \bar{r}_{il} \sqrt{\bar{\mu}_l}\, \bar{s}_{lm} \bar{z}_m, \quad (i = 1, 2, \ldots\ldots, n)$$

Multiplying the corresponding equations in the sets (3′) and (4), member for member, and summing as to i, we have :

$$\sigma\bar{\sigma} \sum_{i=1}^{n} \frac{z_i \bar{z}_i}{\lambda_i} = \sum_{i=1}^{n} \sum_{j,k,l,m} r_{ij}\, \bar{r}_{il} \sqrt{\mu_j} \sqrt{\mu_l}\, s_{jk}\, \bar{s}_{lm}\, z_k\, \bar{z}_m$$

$$= \sum_{j,k,l,m} \left[\sum_{i=1}^{n} r_{ij}\, \bar{r}_{il} \right] \sqrt{\mu_j} \sqrt{\mu_l}\, s_{jk}\, \bar{s}_{lm}\, z_k\, \bar{z}_m$$

But since R is unitary, $\sum_i r_{ij}\, \bar{r}_{il} = \delta_{jl}$ where δ_{jl} is the Kronecker delta whose value is 1 when $j = l$ and 0 when $j \neq l$.

Therefore, the above equation becomes :

$$(5) \qquad \sigma\bar{\sigma} \sum_{i=1}^{n} \frac{z_i \bar{z}_i}{\lambda_i} = \sum_{j,k,m} \mu_j\, s_{jk}\, \bar{s}_{jm}\, z_k\, \bar{z}_m .$$

Now, if we replace every λ_i and μ_j in (5) by λ_{max} and μ_{max}, the left-hand side decreases (at least it does not increase) and the right-hand side increases (at least it does not decrease), and we have :

$$\frac{\sigma\bar{\sigma}}{\lambda_{max}} \sum_i z_i\, \bar{z}_i \leqq \mu_{max} \sum_{j,k,m} s_{jk}\, \bar{s}_{jm}\, z_k\, \bar{z}_m$$

i.e.,

$$\leqq \mu_{max} \sum_{k,\, m} \left[\sum_j s_{jk}\, \bar{s}_{jm} \right] z_k\, \bar{z}_m$$

Again, since S is unitary, $\sum_j S_{jk}\, \bar{S}_{jm} = \delta_{km}$ where δ_{km} is Kronecker delta, so that

$$\frac{\sigma\bar{\sigma}}{\lambda_{max}} \sum_i z_i\, \bar{z}_i \leqq \mu_{max} \sum_k Z_k\, \bar{Z}_k$$

But since $\sum_i z_i\, \bar{z}_i > 0$, $\frac{\sigma\bar{\sigma}}{\lambda_{max}} \leqq \mu_{max}$

or

$$(6) \qquad \sigma\bar{\sigma} \leqq \lambda_{max}\, \mu_{max} .$$

Further, if instead of replacing every λ_i and μ_j in (5) by λ_{max} and μ_{max} we replace them respectively by λ_{min} and μ_{min} the left-hand side of (5) increases (at least it does not decrease) and the right-hand side of (5) decreases (at least it does not increase), and we have

(7) $\sigma \bar{\sigma} \geqq \lambda_{min} \mu_{min}$.

Combining (6) and (7) we have

$$\lambda_{min} \mu_{min} \leqq \sigma \bar{\sigma} \leqq \lambda_{max} \mu_{max},$$

or

$$\begin{aligned} & c_{min} [f(A) \{ f(A) \}^*] \, c_{min} [g(B) \{ g(B) \}^*] \\ & \quad \leqq c [f(A) \, g(B)] \, \bar{c} [f(A) \, g(B)] \\ & \quad \leqq c_{max} [f(A) \{ f(A) \}^*] c_{max} [g(B) \{ g(B) \}^*] \end{aligned}$$

and the result is established.

It is easy to see that the above result is true even when $f(x) = \frac{f_{1(x)}}{f_{2(x)}}$ and $g(x) = \frac{g_{1(x)}}{g_{2(x)}}$ are two rational functions if, of course, $f_2(A) \neq O$ and $g_2(B) \neq O$.

Particular cases of the above theorem. If in the result of the above theorem we put $f(A) = A$ and $g(B) = B$, we have

$$c_{min}(AA^*) \, c_{min}(BB^*) < c(AB)\bar{c}(AB) < c_{max}(AA^*) \, c_{max}(BB^*),$$

a result recently proved by Roy, [3].

If in the result of the above theorem we put $g(B) = I$, whose all the n characteristic roots are unity, we have

$$\begin{aligned} & c_{min}[f(A) \{ f(A) \}^*] \leqq c [f(A)] \, \bar{c} [f(A)] \\ & \qquad \leqq c_{max} [f(A) \{ f(A) \}^*], \end{aligned}$$

a result proved by Ishaq, [2].

If we put $f(A) = A$ and g(B) $= I$ in the result of the above theorem, we have

$$c_{min}(AA^*) < c(A) \, c(A) < c_{max}(AA^*),$$

a result due to Browne, [1].

3. In this section we shall see that in order that

$$
\begin{aligned}
& c_{min}[f(A)\{f(A)\}^*]\, c_{min}[g(B)\{g(B)\}^*] \\
(8) \qquad & = c[f(A)g(B)]\bar{c}[f(A)g(B)] \\
& = c_{max}[f(A)\{f(A)\}^*]\, c_{max}[g(B)\{g(B)\}^*],
\end{aligned}
$$

where $f(A)$ and $g(B)$ are the same as in the theorem proved in §2, the sufficient but not necessary condition is that A and B be both scalar matrices.

To prove the sufficiency part of the condition let $A = pI$ and $B = qI$, where p and q belong to the field of complex numbers.

$\therefore\ f(A) = f(pI) = (\alpha + i\beta)$ I, say;

and $g(B) = g(qI) = (\gamma + i\delta)$ I say.

$$
\begin{aligned}
\text{Therefore, } c[f(A)\{f(A)\}^*] &= c[(\alpha + i\beta)I\,(\alpha - i\beta)I] \\
&= c[(\alpha^2 + \beta^2)I] \\
&= \alpha^2 + \beta^2, \text{ with multiplicity } n;
\end{aligned}
$$

$$
\begin{aligned}
c[g(B)\{g(B)\}^*] &= c[(\gamma + i\delta)I(\gamma - i\delta)I] \\
&= c[(\gamma^2 + \delta^2)I] = \gamma^2 + \delta^2 \text{ with multiplicity } n;
\end{aligned}
$$

$$
\begin{aligned}
\text{and } c[f(A)g(B)] &= c[(\alpha + i\beta)I\,(\gamma + i\delta)I] \\
&= c[(\alpha\gamma - \beta\delta)I + i(\beta\gamma + \alpha\delta)I] \\
&= (\alpha\gamma - \beta\delta) + i(\beta\gamma + \alpha\delta), \text{ with multiplicity } n.
\end{aligned}
$$

Therefore $c[f(A)g(B)]\bar{c}[f(A)g(B)]$

$$
\begin{aligned}
&= [(\alpha\gamma - \beta\delta) + i(\beta\gamma + \alpha\delta)][(\alpha\gamma - \beta\delta) - i(\beta\gamma + \alpha\delta)] \\
&= (\alpha\gamma - \beta\delta)^2 + (\beta\gamma + \alpha\delta)^2 \\
&= (\alpha^2 + \beta^2)(\gamma^2 + \delta^2) \\
&= c_{min}[f(A)\{f(A)\}^*]c_{min}[g(B)\{g(B)\}^*] \\
&= c_{max}\,[f(A)\{f(A)\}^*]c_{max}[g(B)\{g(B)\}^*].
\end{aligned}
$$

In order to prove that the above condition is not necessary, consider the following example:

$$
\text{Let } A = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix},\ B = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix},\ f(x) = 1 - x^2, g(x) = x^2.
$$

$\therefore \quad f(A) = I - A^2 = I$, which is non-singular, and $g(B) = B^2 = O$.

Now $f(A)[\{f(A)\}^*] = I$, so that

$$c_{min}[f(A)\{f(A)\}^*] = c_{max}[f(A)\{f(A)\}^*] = 1\,;$$

and $g(B)\{g(B)\}^* = O$, so that

$$c_{min}[g(B)\{g(B)\}^*] = c_{max}[g(B)\{g(B)\}^*] = 0.$$

Also $f(A)g(B) = (I - A^2)B^2 = 0$, so that

$$c[f(A)g(B)]\bar{c}[f(A)g(B)] = 0$$

Thus equations (8) hold although the matrices A and B are not scalar.

In conclusion, I am thankful to Dr. S. M. Shah for his helpful criticism on this paper.

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has also been found in terms of the elements of A and B. This result contains (2) as a particular case.

In establishing these results we use the well known facts, see [4; p.149], concerning the operator norm

$\| A \| = \max \| Ay \|$ for $\| y \| = 1$, when $\| y \| = \sqrt{\bar{y}'y}$.

In what follows, $R(A) = \max\limits_{(r)} R_r(A) = \max\limits_{(r)} \sum\limits_{s=1}^{n} |a_{rs}|$ and

$T(A) = \max\limits_{(s)} T_s(A) = \max\limits_{(s)} \sum\limits_{r=1}^{n} |a_{rs}|$.

2. <u>Bound theorem for c(A).</u>

<u>Theorem</u> 1. <u>Let c(A) be an arbitrary characteristic root of A. Then</u>

$$|c(A)| \leq \max\left|c\left(\frac{A+\bar{A}'}{2}\right)\right| + \max\left|c\left(\frac{A-\bar{A}'}{2i}\right)\right| . \qquad (4)$$

<u>Proof</u>. Any square matrix $A = (A+\bar{A}')/2 + i(A-\bar{A}')/2i = M+iN$, say, where M and N are Hermitian. Corresponding to an arbitrary characteristic root λ of A, there exists a nonzero column vector x, such that $\lambda x = Ax$. Then

$$\begin{aligned} |\lambda| \; \| x \| &= \| \lambda x \| = \| Ax \| \\ &= \| (M+iN)x \| \\ &\leq \| M+iN \| \; \| x \| , \end{aligned}$$

where for any square matrix B, $\| B \|^2 = \max\limits_{\|y\|=1} \| By \|^2$

$= \max\limits_{\|y\|=1} (By, By) = \max\limits_{\|y\|=1} (\bar{y}'\bar{B}'By) = c_{max}(\bar{B}'B) = c_{max}(B\bar{B}')$,

c_{max} denoting the greatest characteristic root, and in particular if B is Hermitian $\| B \| = \max |c(B)|$.

Since $\| x \| = \sqrt{\bar{x}'x} > 0$, we have

$$|\lambda| \leq \| M + iN \|$$

THE CHARACTERISTIC ROOTS OF THE PRODUCT OF MATRICES I.

1. Introduction.

Let $A = (a_{rs})$ be an arbitrary square matrix of order n, and $c(A)$ a characteristic root of A. Hirsch [5] proved that

$$|c(A)| \leq n. \max_{1 \leq r,s \leq n} |a_{rs}|, \tag{1}$$

and Browne [3] obtained that

$$|c(A)| \leq (R + T)/2, \tag{2}$$

where $R = \max_{(r)} R_r = \max_{(r)} \sum_{s=1}^{n} |a_{rs}|$, $T = \max_{(s)} T_s = \max_{(s)} \sum_{r=1}^{n} |a_{rs}|$.

Further, Browne [2] showed that

$$s \leq |c(A)|^2 \leq G, \tag{3}$$

where s and G are the smallest and the greatest characteristic roots of $A\bar{A}'$.

(1), (2) and other results established by several authors (for a list of references see [1]) give upper limits to $|c(A)|$ in terms of the elements of A, while (3) gives limits to $|c(A)|$ in terms of the characteristic roots of $A\bar{A}'$. In this paper an upper limit for $|c(A)|$ has been found in terms of the characteristic roots of the associated Hermitian matrices $(A + \bar{A}')/2$ and $(A - \bar{A}')/2i$, and this result has been extended to the characteristic roots of the product matrix AB of two n-square real or complex matrices A and B. An upper limit for $|c(AB)|$

$$\leq \quad \| M \| + \| N \|$$
$$= \max|c(\tfrac{A+\bar{A}'}{2})| + \max|c(\tfrac{A-\bar{A}'}{2i})| .$$

This establishes (4).

It may be observed that (3) and (4) give the same upper limit for $|c(A)|$ when A is Hermitian or skew-Hermitian. There are some other matrices also, for example $\begin{bmatrix} 1 & i \\ -3i & -1 \end{bmatrix}$, which are not of some special type but for which (3) and (4) give the same upper limit for $|c(A)|$. The limit for $|c(A)|$ given by (4) is not better than that given by (3), but, in general, it is easier to find the characteristic roots of $(A+\bar{A}')/2$ and $(A-\bar{A}')/2i$ than those of $A\bar{A}'$.

3. Bound theorems for c(AB).

We now consider two theorems which give upper limits to the absolute value of an arbitrary characteristic root of the product matrix AB.

Theorem 2. Let c(AB) denote an arbitrary characteristic root of AB. Then

$$|c(AB)| \leq \left[\max|c(\tfrac{A+\bar{A}'}{2})| + \max|c(\tfrac{A-\bar{A}'}{2i})|\right]\left[\max|c(\tfrac{B+\bar{B}'}{2})| + \max|c(\tfrac{B-\bar{B}'}{2i})|\right]. \qquad (5)$$

Proof. Any square matrix $A = (A+\bar{A}')/2 + i(A-\bar{A}')/2i = M + iN$, say, and $B = (B+\bar{B}')/2 + i(B-\bar{B}')/2i = R + iS$, say, where M, N, R ans S are Hermitian. Thus $AB = (M+iN)(R+iS)$.

If μ is an arbitrary characteristic root of AB, there exists a nonzero column vector x, such that $\mu x = ABx$ so that

$$|\mu| \, \| x \| = \| \mu x \| = \| ABx \|$$
$$\leq \| M+iN \| \, \| R+iS \| \, \| x \|$$

Since $\| x \| > 0$, we have

$$\begin{aligned} |\mu| &\le \| M+iN \| \, \| R+iS \| \\ &\le (\| M \| + \| N \|)(\| R \| + \| S \|) \\ &= (\max|c(M)|+\max|c(N)|)(\max|c(R)|+\max|c(S)|), \end{aligned}$$

since M, N, R, and S are Hermitian matrices.

This establishes (5). It is easy to see that (5) can be generalised to the case of the product of any finite number of matrices $A_1, A_2, \ldots, A_k$ of the same order. The result in the generalised form is

$$\left|c\left(\prod_{\gamma=1}^{k} A_r\right)\right| \le \prod_{\gamma=1}^{k} \left\{ \max\left|c\left(\frac{A_\nu+\bar{A}_\nu'}{2}\right)\right| + \max\left|c\left(\frac{A_\nu-\bar{A}_\nu'}{2i}\right)\right| \right\}. \tag{6}$$

<u>Theorem</u> 3. <u>Let $c(AB)$ denote an arbitrary characteristic root of AB. Then</u>

$$|c(AB)| \le \frac{R(A) + T(A)}{2} \; \frac{R(B) + T(B)}{2}. \tag{7}$$

<u>Proof.</u> If μ is an arbitrary characteristic root of AB, there exists a nonzero column vector x, such that $\mu x = ABx$.

This gives, as we have seen in the proof of the previous theorem,

$$\begin{aligned} |\mu| &\le \| AB \| \le \| A \| \, \| B \| \\ &\le \frac{R(A)+T(A)}{2} \cdot \frac{R(B)+T(B)}{2}, \end{aligned}$$

for, by a known result [3; p.703]

$$\| A \| = \left| c_{\max}^{1/2}(A\bar{A}') \right| \le \frac{R(A)+T(A)}{2}.$$

This completes the proof of Theorem 3.

It may be observed that (7) can be generalised to the case of the product of a finite number of matrices $A_1, A_2, \ldots, A_k$

of the same order. The result in the generalised form is

$$\left| c\left(\prod_{r=1}^{k} A_r \right) \right| \leq \prod_{r=1}^{k} \frac{R(A_r) + T(A_r)}{2} . \tag{8}$$

Particular cases of Theorem 3.

(i) If we put $B = I$, (7) reduces to

$$|c(A)| \leq \frac{R(A) + T(A)}{2},$$

a result due to Browne [3], quoted earlier.

(ii) If A and B are Hermitian or skew-Hermitian matrices, then $R_r(A) = T_r(A)$ and $R_r(B) = T_r(B)$ for $r = 1,2,\ldots,n$, which means $R(A) = T(A)$ and $R(B) = T(B)$. Hence for Hermitian or skew-Hermitian matrices A and B

$$|c(AB)| \leq R(A)R(B). \tag{9}$$

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THE CHARACTERISTIC ROOTS OF THE PRODUCT OF MATRICES

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1. Introduction

DURING the last fifty or sixty years several authors† have given limits to the characteristic roots of an arbitrary matrix $A = (a_{ij})$. In short, Browne [(**3**) 263] showed in 1930 that

$$|c(A)| \leqslant \tfrac{1}{2}(R+T), \tag{1}$$

where

$$R = \max_{(i)} R_i = \max_{(i)} \sum_j |a_{ij}|, \qquad T = \max_{(j)} T_j = \max_{(j)} \sum_i |a_{ij}|,$$

and $c(A)$ denotes a characteristic root of A. This result of Browne's was improved by Parker (**5**) in 1937 who showed that

$$|c(A)| \leqslant \max_{(i)}\{\tfrac{1}{2}(R_i+T_i)\}. \tag{2}$$

The inequality (1) was further sharpened by Farnell (**4**) in 1944 when he obtained

$$|c(A)| \leqslant (RT)^{\frac{1}{2}}, \tag{3}$$

and was still further improved by Barankin (**1**) in 1945, who proved that

$$|c(A)| \leqslant \max_{(i)}(R_i T_i)^{\frac{1}{2}}. \tag{4}$$

It is the purpose of this paper to give an alternative proof of (3) and to extend it to the characteristic roots of the product of two n-square matrices $A = (a_{ij})$ and $B = (b_{ij})$ whose elements lie in the field of complex numbers. In what follows the sum of the absolute values of the elements in the ith row of A will be denoted by $R_i(A)$, the sum of the absolute values of the elements of A in the jth column by $T_j(A)$, and $R(A)$ and $T(A)$ will stand respectively for the maxima of the $R_i(A)$ and of the $T_j(A)$.

† References are given in (**3**) and (**6**).

2. The theorem given by Farnell may be stated as follows:

FARNELL'S THEOREM. *Let* $R_r = \sum_s |a_{rs}|$, $T_s = \sum_r |a_{rs}|$, $R = \max(R_r)$, *and* $T = \max(T_s)$. *Then* $|c(A)| \leqslant (RT)^{\frac{1}{2}}$.

To prove this let $\lambda_1^2 \leqslant \lambda_2^2 \leqslant \ldots \leqslant \lambda_n^2$

be the (necessarily real) characteristic roots of the Hermitian matrix $A\bar{A}'$. Then there exists a unitary matrix $U = (u_{ij})$, such that

$$A\bar{A}' = UD(\lambda^2)\bar{U}',$$

where $$D(\lambda^2) = \operatorname{diag}(\lambda_1^2, \lambda_2^2, \ldots, \lambda_n^2),$$

i.e. $$D(\lambda^2) = \bar{U}'(A\bar{A}')U,$$

so that $$\lambda_i^2 = \sum_{r,s} \bar{u}_{ri}\, \alpha_{rs}\, u_{si},$$

where α_{rs} stands for the element of $A\bar{A}'$ in the (r,s)th position. This equation may be put as

$$\lambda_i^2 = \sum_{r,s} \alpha_{rs}\, \bar{x}_r\, x_s, \tag{5}$$

where $x_1, x_2, \ldots, x_n$ is a set of complex numbers such that $\sum_r x_r \bar{x}_r = 1$.

Let $|x_r| = \xi_r$, so that $\sum_r \xi_r^2 = 1$.

Taking the absolute value of (5) and remembering that λ_i's are all real, we have

$$\begin{aligned}
\lambda_i^2 &\leqslant \sum_{r,s} |\alpha_{rs}| \xi_r \xi_s \\
&\leqslant \tfrac{1}{2} \sum_{r,s} |\alpha_{rs}| (\xi_r^2 + \xi_s^2) \\
&= \tfrac{1}{2}\Big(\sum_{r,s} |\alpha_{rs}| \xi_r^2 + \sum_{r,s} |\alpha_{rs}| \xi_s^2 \Big) \\
&= \tfrac{1}{2}\Big(\sum_r \xi_r^2 \sum_s |\alpha_{rs}| + \sum_s \xi_s^2 \sum_r |\alpha_{rs}| \Big) \\
&= \tfrac{1}{2}\Big\{ \sum_r \xi_r^2 R_r(A\bar{A}') + \sum_s \xi_s^2 T_s(A\bar{A}') \Big\} \\
&= \tfrac{1}{2}\Big\{ \sum_r \xi_r^2 R_r(A\bar{A}') + \sum_r \xi_r^2 T_r(A\bar{A}') \Big\} \\
&= \sum_r \xi_r^2 R_r(A\bar{A}'),
\end{aligned}$$

since, $A\bar{A}'$ being Hermitian, $T_r(A\bar{A}') = R_r(A\bar{A}')$. Suppose that $R_r(A\bar{A}')$ attains its maximum value for $r = k$; then

$$\lambda_i^2 \leqslant R_k(A\bar{A}').$$

But
$$R_k(A\bar{A}') = \left|\sum_s a_{ks}\bar{a}_{1s}\right| + \left|\sum_s a_{ks}\bar{a}_{2s}\right| + \ldots + \left|\sum_s a_{ks}\bar{a}_{ns}\right|$$
$$\leqslant \sum_s |a_{ks}||\bar{a}_{1s}| + \sum_s |a_{ks}||\bar{a}_{2s}| + \ldots + \sum_s |a_{ks}||\bar{a}_{ns}|$$
$$= |a_{k1}|\sum_t |a_{t1}| + |a_{k2}|\sum_t |a_{t2}| + \ldots + |a_{kn}|\sum_t |a_{tn}| \tag{6}$$
$$= |a_{k1}|T_1(A) + |a_{k2}|T_2(A) + \ldots + |a_{kn}|T_n(A)$$
$$\leqslant T(A)R_k(A)$$
$$\leqslant R(A)T(A),$$
so that
$$\lambda_i^2 \leqslant R(A)T(A) \quad (i = 1, 2, \ldots, n).$$

But, by a well-known result of Browne's [(3) 263]
$$c(A)\bar{c}(A) \leqslant c_{\max}(A\bar{A}') = \lambda_n^2,$$
i.e.
$$|c(A)|^2 \leqslant \lambda_n^2 \leqslant R(A)T(A).$$
Accordingly
$$|c(A)| \leqslant \{R(A)T(A)\}^{\frac{1}{2}}.$$

I now prove the following theorem which gives the upper bound for the modulus of an arbitrary characteristic root of the product of two n-square matrices A and B.

THEOREM 1. *Let $c(AB)$ denote an arbitrary characteristic root of AB. Then*
$$|c(AB)| \leqslant \{R(A)T(A)R(B)T(B)\}^{\frac{1}{4}}.$$

Proof. Let $\lambda_1^2 \leqslant \lambda_2^2 \leqslant \ldots \leqslant \lambda_n^2$ be the (necessarily real) characteristic roots of the Hermitian matrix $A\bar{A}'$. Then there exists a unitary matrix $P = (p_{ij})$ such that
$$D(\lambda^2) = \bar{P}'(A\bar{A}')P, \tag{7}$$
where $D(\lambda^2) = \mathrm{diag}(\lambda_1^2, \lambda_2^2, \ldots, \lambda_n^2)$.

Similarly there exists a unitary matrix $Q = (q_{ij})$ such that
$$D(\mu^2) = \bar{Q}'(B\bar{B}')Q, \tag{8}$$
where
$$D(\mu^2) = \mathrm{diag}(\mu_1^2, \mu_2^2, \ldots, \mu_n^2)$$
and $\mu_1^2 \leqslant \mu_2^2 \leqslant \ldots \leqslant \mu_n^2$ are the (necessarily real) characteristic roots of $B\bar{B}'$.

From (7) and (8) we obtain
$$D(\lambda^2\mu^2) = \bar{P}'A\bar{A}'P\bar{Q}'B\bar{B}'Q,$$
i.e.
$$\lambda_i^2\mu_i^2 = \sum_{r,s,t,u,v} \bar{p}_{ri}\alpha_{rs}p_{st}\bar{q}_{ut}\beta_{uv}q_{vi},$$
where α_{rs} and β_{uv} denote the (r,s)th and (u,v)th elements of $A\bar{A}'$ and $B\bar{B}'$ respectively.

Since P and Q are Hermitian matrices, the above equation may be written as

$$\lambda_i^2 \mu_i^2 = \sum_{r,s,t} \alpha_{rs} \bar{x}_r x_s \beta_{st} \bar{y}_s y_t, \tag{9}$$

where $x_1, x_2,..., x_n$ and $y_1, y_2,..., y_n$ are sets of complex numbers, not all zero, such that $\sum_r x_r \bar{x}_r = \sum_s y_s \bar{y}_s = 1$.

Let $|x_r| = \xi_r$ and $|y_s| = \eta_s$, so that

$$\sum_r \xi_r^2 = \sum_s \eta_s^2 = 1.$$

On taking the absolute values of (9), we have

$$\begin{aligned}
\lambda_i^2 \mu_i^2 &\leqslant \sum_{r,s,t} |\alpha_{rs}| \xi_r \xi_s |\beta_{st}| \eta_s \eta_t \\
&\leqslant \sum_{r,s} |\alpha_{rs}| \xi_r \xi_s \sum_{s,t} |\beta_{st}| \eta_s \eta_t \\
&\leqslant \tfrac{1}{4}\Big\{ \sum_{r,s} |\alpha_{rs}| (\xi_r^2 + \xi_s^2) \sum_{s,t} |\beta_{st}| (\eta_s^2 + \eta_t^2) \Big\} \\
&= \tfrac{1}{4}\Big\{ \sum_r \xi_r^2 R_r(A\bar{A}') + \sum_s \xi_s^2 T_s(A\bar{A}') \Big\}\Big\{ \sum_s \eta_s^2 R_s(B\bar{B}') + \sum_t \eta_t^2 T_t(B\bar{B}') \Big\} \\
&\leqslant \Big\{ \sum_r R_r(A\bar{A}') \xi_r^2 \Big\}\Big\{ \sum_s R_s(B\bar{B}') \eta_s^2 \Big\}, \\
&\qquad \text{since } R_r(A\bar{A}') = T_r(A\bar{A}') \text{ and } R_s(B\bar{B}') = T_s(B\bar{B}'), \\
&\leqslant R_k(A\bar{A}') R_l(B\bar{B}'),
\end{aligned}$$

if $R_r(A\bar{A}')$ and $R_s(B\bar{B}')$ attain their maximum values for $r = k$ and $s = l$ respectively.

But, as proved in (6),

$$R_k(A\bar{A}') \leqslant R(A)T(A).$$

Similarly $$R_l(B\bar{B}') \leqslant R(B)T(B).$$

Thus we have

$$\lambda_i^2 \mu_i^2 \leqslant R(A)T(A)R(B)T(B), \quad \text{which is true for } i = 1, 2,..., n.$$

By a recently proved result given by Roy (**7**),

$$c(AB)\bar{c}(AB) \leqslant c_{\max}(A\bar{A}')c_{\max}(B\bar{B}') = \lambda_n^2 \mu_n^2.$$

Hence, $$|c(AB)|^2 \leqslant R(A)T(A)R(B)T(B). \tag{10}$$

This completes the proof of the theorem. It may be observed that by putting $B = I$ in (10) we get the inequality (3).

I now establish another theorem which, in general, gives a better upper bound for $|c(AB)|$ than that given by (10).

THEOREM 2. *Let $c(AB)$ denote an arbitrary characteristic root of AB. Then*
$$|c(AB)| \leqslant \min\{R(A)R(B),\, T(A)T(B)\},$$
i.e. the absolute value of an arbitrary characteristic root of AB does not exceed the smaller of the numbers $R(A)R(B)$ and $T(A)T(B)$.

Proof. If λ denotes an arbitrary characteristic root of
$$AB = \Big(\sum_r a_{ir} b_{rj}\Big),$$
there exists a set of complex numbers $x_1, x_2,..., x_n$, not all zero, such that
$$\lambda x_i = \sum_j \sum_r a_{ir} b_{rj} x_j \quad (i = 1, 2,..., n) \tag{11}$$
and a set of complex numbers $y_1, y_2,..., y_n$, not all zero, such that
$$\lambda y_i = \sum_j \sum_r a_{jr} b_{ri} y_j \quad (i = 1, 2,..., n). \tag{12}$$

Let $|x_i| = \xi_i$ and let ξ_k be the greatest of the ξ_i; then from
$$\lambda x_k = \sum_j \sum_r a_{kr} b_{rj} x_j$$
it follows that
$$|\lambda x_k| = \Big|\sum_{j,r} a_{kr} b_{rj} x_j\Big|,$$
i.e.
$$|\lambda|\xi_k \leqslant \sum_{j,r} |a_{kr}||b_{rj}|\xi_j \leqslant \Big\{\sum_{j,r} |a_{kr}||b_{rj}|\Big\}\xi_k.$$

But, since $\xi_k \neq 0$, we have
$$\begin{aligned} |\lambda| &\leqslant \sum_r |a_{kr}||b_{r1}| + \sum_r |a_{kr}||b_{r2}| + \dots + \sum_r |a_{kr}||b_{rn}| \\ &= |a_{k1}| \sum_s |b_{1s}| + |a_{k2}| \sum_s |b_{2s}| + \dots + |a_{kn}| \sum_s |b_{ns}| \\ &= |a_{k1}| R_1(B) + |a_{k2}| R_2(B) + \dots + |a_{kn}| R_n(B) \\ &\leqslant R_k(A) R(B), \end{aligned}$$
and hence
$$|\lambda| \leqslant R(A)R(B). \tag{13}$$

Now let $|y_i| = \eta_i$ and let η_l be the greatest of the η_i, so that
$$\lambda y_l = \sum_{j,r} a_{jr} b_{rl} y_j.$$

Taking the absolute value of the above equation and proceeding as we did in establishing (13), we obtain
$$|\lambda| \leqslant T(A)T(B). \tag{14}$$
Combining (13) and (14) we establish the theorem.

In particular, if we put $B = I$ in the result of Theorem 2, we have
$$|c(A)| \leqslant \min\{R(A),\, T(A)\},$$

a result due to Frobenius [(**2**) 387] for a matrix A with positive elements, and due to Brauer (**2**) for an arbitrary matrix A.

It may be remarked that the limits given in Theorem 2 are, in general, better than those given by Theorem 1. However, when A and B are such that

$$R(A)R(B) = T(A)T(B),$$

then the two theorems give the same upper bound for $|c(AB)|$. In particular, if A and B are Hermitian matrices, then $R(A) = T(A)$ and $R(B) = T(B)$, so that $|c(AB)| \leqslant R(A)R(B)$. Thus we have

THEOREM 3. *Let A and B be two n-square Hermitian matrices and $c(AB)$ stand for an arbitrary characteristic root of AB. Then*

$$|c(AB)| \leqslant R(A)R(B).$$

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THE CHARACTERISTIC ROOTS OF THE PRODUCT OF MATRICES III.

1. Introduction.

In papers vi and vii the upper limits for an arbitrary characteristic root, $c(AB)$, of AB have been found out in terms of the characteristic roots of the associated Hermitian matrices $(A+\bar{A}')/2$, $(A-\bar{A}')/2i$, $(B+\bar{B}')/2$ and $(B-\bar{B}')/2i$ and in terms of the elements of A and B. It is the purpose of this paper to find the upper limits for the real and imaginary parts of $c(AB)$ in terms of the elements of $(A+\bar{A}')/2$, $(A-\bar{A}')/2i$, $(B+\bar{B}')/2$ and $(B-\bar{B}')/2i$. In what follows, $R_i(A)$ will denote the sum of the absolute values of the elements of an arbitrary matrix A in the i-th row, $T_i(A)$ will denote the sum of the absolute values of the elements of A in the i-th column, and $R(A)$, $T(A)$ will stand, respectively, for the greatest of the $R_i(A)$ and the $T_i(A)$.

2. Upper bounds for the real and imaginary parts of $c(AB)$.

Theorem. Let A and B be two commuting n-square complex matrices. If $S_r'(A)$, $S_r''(A)$, $S_r'(B)$, $S_r''(B)$ are the sums of the absolute values of the elements in the r-th row of $(A+\bar{A}')/2$, $(A-\bar{A}')/2i$, $(B+\bar{B}')/2$, $(B-\bar{B}')/2i$, respectively, and if $S'(A)$, $S''(A)$, $S'(B)$, $S''(B)$ are, respectively, the greatest of the $S_r'(A)$, $S_r''(A)$, $S_r'(B)$, $S_r''(B)$, then

$$\left|\frac{c(AB)+\bar{c}(AB)}{2}\right| \le S'(A)S'(B) + S''(A)S''(B), \qquad (1)$$

and $$\left|\frac{c(AB)-\bar{c}(AB)}{2i}\right| \le S'(A)S''(B) + S''(A)S'(B). \qquad (2)$$

Proof. Any square matrix $A = (A+\bar{A}')/2 + i(A-\bar{A}')/2i$ $= P + iQ$, say, where $P = (p_{ij})$, $Q = (q_{ij})$ are Hermitian matrices; and any square matrix $B = (B+\bar{B}')/2 + i(B-\bar{B}')/2i = U + iV$, say, where $U = (u_{ij})$ and $V = (v_{ij})$ are Hermitian matrices. Thus

$$AB = PU - QV + i(PV + QU), \qquad (3)$$

and $$\bar{A}'\bar{B}' = PU - QV - i(PV + QU). \qquad (4)$$

Now, if μ is a characteristic root of AB, there exists a complex unit vector $x = (x_1, x_2, \ldots, x_n)^T$, such that

$$\mu x = ABx.$$

Premultiplying the above equation by $\bar{x}'$, we have

$$\mu\bar{x}'x = \bar{x}'ABx,$$

or, $$\mu = \bar{x}'ABx. \qquad (5)$$

Taking the conjugate transpose of (5), we have

$$\bar{\mu} = \bar{x}'(\bar{B}'\bar{A}')x$$

$$= \bar{x}'(\bar{A}'\bar{B}')x, \qquad (6)$$

since $AB = BA$ implies $\bar{A}'\bar{B}' = \bar{B}'\bar{A}'$.

From (5) and (6) by addition and subtraction, we have

$$(\mu + \bar{\mu})/2 = \bar{x}'(PU - QV)x, \qquad (7)$$

and $$(\mu - \bar{\mu})/2i = \bar{x}'(PV + QU)x. \qquad (8)$$

From (7) and (8) we determine the upper bounds for $|(\mu+\bar{\mu})/2|$ and $|(\mu - \bar{\mu})/2i|$. Since these relations are identical in form, it is sufficient to carry the computation through one of them only.

Taking the absolute values in (7), we get

$$\left|(\mu + \bar{\mu})/2\right| = \left|\bar{x}'(PU-QV)x\right|$$

$$= \left|\sum_{r,s=1}^{n}\alpha_{rs}\bar{x}_r x_s - \sum_{r,s=1}^{n}\beta_{rs}\bar{x}_r x_s\right| ,$$

where α_{rs} and β_{rs} denote the elements of PU and QV, respectively, in the (r,s)-th position.

$$\text{or,} \quad \left|\frac{c(AB)+\bar{c}(AB)}{2}\right| \leq \left|\sum_{r,s=1}^{n}\alpha_{rs}\bar{x}_r x_s\right| + \left|\sum_{r,s=1}^{n}\beta_{rs}\bar{x}_r x_s\right| . \tag{9}$$

Let $\zeta_r = |x_r|$, so that $\sum_{r=1}^{n}\zeta_r^2 = 1$ and $\zeta_r\zeta_s \leq \frac{1}{2}(\zeta_r^2 + \zeta_s^2)$.

Now, we consider the two terms on the right-hand side of (9) separately.

$$\left|\sum_{r,s=1}^{n}\alpha_{rs}\bar{x}_r x_s\right| \leq \sum_{r,s=1}^{n}|\alpha_{rs}|\zeta_r\zeta_s$$

$$\leq \frac{1}{2}\sum_{r,s=1}^{n}|\alpha_{rs}|(\zeta_r^2 + \zeta_s^2)$$

$$= \frac{1}{2}\left[\sum_{r=1}^{n}\zeta_r^2\sum_{s=1}^{n}|\alpha_{rs}| + \sum_{s=1}^{n}\zeta_s^2\sum_{r=1}^{n}|\alpha_{rs}|\right]$$

$$\leq \frac{1}{2}\left[\sum_{r=1}^{n}\zeta_r^2 R_r(PU) + \sum_{s=1}^{n}\zeta_s^2 T_s(PU)\right] .$$

Supposing that $R_r(PU)$ and $T_s(PU)$ attain their maximum values, respectively, for r=h and s=k, we have

$$\left|\sum_{r,s=1}^{n}\alpha_{rs}\bar{x}_r x_s\right| \leq \frac{1}{2}\left[R_h(PU) + T_k(PU)\right] .$$

But, by definition,

$$R_h(PU) = \left|\sum_s p_{hs}u_{s1}\right| + \left|\sum_s p_{hs}u_{s2}\right| + \dots + \left|\sum_s p_{hs}u_{sn}\right|$$

$$\leq \sum_s|p_{hs}||u_{s1}| + \sum_s|p_{hs}||u_{s2}| + \dots + \sum_s|p_{hs}||u_{sn}|$$

$$= |p_{h1}|\sum_t|u_{1t}| + |p_{h2}|\sum_t|u_{2t}| + \dots + |p_{hn}|\sum_t|u_{nt}|$$

$$= |p_{h1}|R_1(U) + |p_{h2}|R_2(U) + \dots + |p_{hn}|R_n(U)$$

$$\leq R(U)\left[|p_{h1}| + |p_{h2}| + \dots + |p_{hn}|\right]$$

$$= R_h(P)R(U)$$

$$\leq R(P)R(U) = S'(A)S'(B); \tag{10}$$

and $T_k(PU) = |\sum_s p_{1s}u_{sk}| + |\sum_s p_{2s}u_{sk}| + \dots + |\sum_s p_{ns}u_{sk}|$

$$\leq \sum_s |p_{1s}||u_{sk}| + \sum_s |p_{2s}||u_{sk}| + \dots + \sum_s |p_{ns}||u_{sk}|$$

$$= |u_{1k}|\sum_t |p_{t1}| + |u_{2k}|\sum_t |p_{t2}| + \dots + |u_{nk}|\sum_t |p_{tn}|$$

$$= |u_{1k}|T_1(P) + |u_{2k}|T_2(P) + \dots + |u_{nk}|T_n(P)$$

$$\leq T(P)T_k(U)$$

$$\leq T(P)T(U) = S'(A)S'(B), \tag{11}$$

since for any Hermitian matrix $H = (h_{rs})$, $T(H) = \max_{(s)} T_s(H)$
$= \max_{(s)} \sum_{r=1}^{n} |h_{rs}| = \max_{(s)} \sum_{r=1}^{n} |h_{sr}| = \max_{(s)} R_s(H) = R(H)$.

The inequalities (10) and (11) give

$$|\sum_{r,s} \alpha_{rs}\bar{x}_r x_s| \leq S'(A)S'(B). \tag{12}$$

Similarly, taking $|\sum_{r,s} \beta_{rs}\bar{x}_r x_s|$ and proceeding as we did in establishing (12), we shall have

$$|\sum_{r,s} \beta_{rs}\bar{x}_r x_s| \leq R(Q)R(V) = S''(A)S''(B). \tag{13}$$

Combining (12) and (13), we ob tain

$$\left|\frac{c(AB) + \bar{c}(AB)}{2}\right| \leq S'(A)S'(B) + S''(A)S''(B).$$

Similarly, starting with (8), we can establish the inequality (2).

This completes the proof of the theorem.

The condition, that A and B commute, imposed on the matrices in the Theorem, is necessary as shown by the following Example:

$$A = \begin{bmatrix} 0 & i \\ 2i & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & -2i \\ -i & 0 \end{bmatrix}, \quad AB = \begin{bmatrix} 1 & 0 \\ 0 & 4 \end{bmatrix} \neq BA.$$

Here $S'(A) = S'(B) = 1/2$, $S''(A) = S''(B) = 3/2$, $c(AB) = 1, 4$, and 4 is not less than or equal to 5/2.

3. <u>Some particular cases of (1) and (2).</u>

(i). Let A and B be commuting n-square Hermitian matrices, so that AB is also Hermitian and all $c(AB)$ are real. In this case $S'(A) = R(A)$, $S'(B) = R(B)$, and $S''(A) = S''(B) = 0$. Thus, for matrices A and B defined above, (1) reduces to

$$|c(AB)| \leq R(A)R(B), \tag{14}$$

a result proved in [2] .

(ii). Again, if A and B are skew-Hermitian matrices of the same order, $A+\bar{A}' = B+\bar{B}' = 0$, $(A-\bar{A}')/2i = A/i$, and $(B-\bar{B}')/2i = B/i$. Also AB is Hermitian, so that all the characteristic roots of AB are real, $S'(A) = S'(B) = 0$ and $S''(A) = R(A/i) = R(A)$, and $S''(B) = R(B/i) = R(B)$. In this case also (1) reduces to

$$|c(AB)| \leq R(A)R(B). \tag{15}$$

(iii). Let us put B = I, for which $S'(B) = 1$ and $S''(B) = 0$. In this case (1) and (2) reduce to

$$\left|\frac{c(A)+\bar{c}(A)}{2}\right| \leq \text{[illegible]}, \tag{16}$$

$$\text{and} \quad \left|\frac{c(A) - \bar{c}(A)}{2i}\right| \leq S''(A), \tag{17}$$

results due to Browne [1] and Parker [3], giving the upper bounds for the real and imaginary parts of an arbitrary characteristic root of A in terms of the elements of the associated Hermitian matrices $(A+\bar{A}')/2$ and $(A-\bar{A}')/2i$.

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THE CHARACTERISTIC ROOTS OF THE PRODUCT OF MATRICES IV.

1. Introduction.

Let A and B be two n-square matrices with elements belonging to the field of complex numbers. In the previous paper the limits for the real and imaginary parts of $c(AB)$ have been found in terms of the elements of the four auxiliary matrices $(A+\bar{A}')/2$, $(A-\bar{A}')/2i$, $(B+\bar{B}')/2$ and $(B-\bar{B}')/2i$. The object of this paper is to find the lower and upper limits for the real and imaginary parts of $c(AB)$ in terms of the characteristic roots of the above four Hermitian matrices. The results proved here contain as special cases the results

$$c_{min}\left(\frac{A+\bar{A}'}{2}\right) \leq \frac{c(A)+\bar{c}(A)}{2} \leq c_{max}\left(\frac{A+\bar{A}'}{2}\right),$$

and
$$c_{min}\left(\frac{A-\bar{A}'}{2i}\right) \leq \frac{c(A)-\bar{c}(A)}{2i} \leq c_{max}\left(\frac{A-\bar{A}'}{2i}\right)$$

due to Hirsch [2] and Bromwich [1] , which give the lower and upper limits for the real and imaginary parts of $c(A)$ in terms of the characteristic roots of $(A+\bar{A}')/2$ and $(A-\bar{A}')/2i$.

2. The bound theorems for $c(AB)$.

Theorem 1. Let A and B be two commuting n-square complex matrices, such that the Hermitian matrices $(A+\bar{A}')/2, (A-\bar{A}')/2i$, $(B+\bar{B}')/2$ and $(B-\bar{B}')/2i$ are at least positive semi-definite.

Then

$$c_{min}\left(\frac{A+\bar{A}'}{2}\right)c_{min}\left(\frac{B+\bar{B}'}{2}\right) - c_{max}\left(\frac{A-\bar{A}'}{2i}\right)c_{max}\left(\frac{B-\bar{B}'}{2i}\right)$$

$$\leq \frac{c(AB) + \bar{c}(AB)}{2} \tag{1}$$

$$\leq c_{max}\left(\frac{A+\bar{A}'}{2}\right)c_{max}\left(\frac{B+\bar{B}'}{2}\right) - c_{min}\left(\frac{A-\bar{A}'}{2i}\right)c_{min}\left(\frac{B-\bar{B}'}{2i}\right),$$

and

$$c_{min}\left(\frac{A+\bar{A}'}{2}\right)c_{min}\left(\frac{B-\bar{B}'}{2i}\right) + c_{min}\left(\frac{A-\bar{A}'}{2i}\right)c_{min}\left(\frac{B+\bar{B}'}{2}\right)$$

$$\leq \frac{c(AB) - \bar{c}(AB)}{2i} \tag{2}$$

$$\leq c_{max}\left(\frac{A+\bar{A}'}{2}\right)c_{max}\left(\frac{B-\bar{B}'}{2i}\right) + c_{max}\left(\frac{A-\bar{A}'}{2i}\right)c_{max}\left(\frac{B+\bar{B}'}{2}\right),$$

where c_{max} and c_{min} stand, respectively, for the greatest and least characteristic roots.

Proof. Any square matrix $A = (A+\bar{A}')/2 + (A - \bar{A}')/2$ where $(A+\bar{A}')/2$ is Hermitian and $(A-\bar{A}')/2$ is skew-Hermitian. Let the characteristic roots (necessarily real but not necessarily distinct) of $(A + \bar{A}')/2$ be $\lambda_1, \lambda_2, \ldots, \lambda_n$ arranged in any order. Then there exists a unitary matrix P, such that

$$\bar{P}'\left(\frac{A+\bar{A}'}{2}\right)P = \text{diag.}(\lambda_1, \lambda_2, \ldots, \lambda_n) \equiv D(\lambda), \text{ say,}$$

whence

$$\left(\frac{A+\bar{A}'}{2}\right) = PD(\lambda)\bar{P}'. \tag{3}$$

Since the matrix $(A-\bar{A}')/2i$ is also Hermitian, there exists a unitary matrix Q, such that

$$\bar{Q}'\left(\frac{A-\bar{A}'}{2i}\right)Q = \text{diag.}(\gamma_1, \gamma_2, \ldots, \gamma_n) \equiv D(\gamma), \text{ say,}$$

where $\gamma_1, \gamma_2, \ldots, \gamma_n$ are the (real) characteristic roots of $(A - \bar{A}')/2i$.

Therefore

$$\left(\frac{A-\bar{A}'}{2i}\right) = QD(\gamma)\bar{Q}'. \tag{4}$$

Equations (3) and (4) give

$$A = PD(\lambda)\bar{P}' + iQD(\gamma)\bar{Q}'. \tag{5}$$

Similarly, let $\mu_1, \mu_2, \dots, \mu_n$ be the characteristic roots (necessarily real) of $(B+\bar{B}')/2$, $\nu_1, \nu_2, \dots, \nu_n$ be the real characteristic roots of $(B-\bar{B}')/2i$, so that there exist unitary matrices R and S such that

$$B = RD(\mu)\bar{R}' + iSD(\nu)\bar{S}', \tag{6}$$

where $D(\mu) \equiv \text{diag.}(\mu_1, \mu_2, \dots, \mu_n)$ and $D(\nu) \equiv \text{diag.}(\nu_1, \nu_2, \dots, \nu_n)$.

From (5) and (6) we have

$$AB = PD(\lambda)\bar{P}'RD(\mu)\bar{R}' - QD(\gamma)\bar{Q}'SD(\nu)\bar{S}' + i\left[PD(\lambda)\bar{P}'SD(\nu)\bar{S}' + QD(\gamma)\bar{Q}'RD(\mu)\bar{R}'\right]; \tag{7}$$

and

$$\bar{A}'\bar{B}' = PD(\lambda)\bar{P}'RD(\mu)\bar{R}' - QD(\gamma)\bar{Q}'SD(\nu)\bar{S}' - i\left[PD(\lambda)\bar{P}'SD(\nu)\bar{S}' + QD(\gamma)\bar{Q}'RD(\mu)\bar{R}'\right]. \tag{8}$$

Now, if σ is a characteristic root of AB, there exists a non-null column vector $x = (x_1, x_2, \dots, x_n)$, with complex coordinates, such that

$$\sigma x = ABx$$

or, $$\sigma \bar{x}'x = \bar{x}'ABx. \tag{9}$$

Taking the conjugate transpose of both the sides of (9), we have

$$\bar{\sigma}\,\bar{x}'x = \bar{x}'(\bar{B}'\bar{A}')x = \bar{x}'(\bar{A}'\bar{B}')x, \tag{10}$$

since AB = BA implies that $\bar{B}'\bar{A}' = \bar{A}'\bar{B}'$.

Adding (9) and (10), we have

$$(\sigma+\bar{\sigma})\bar{x}'x = \bar{x}'\left[AB + \bar{A}'\bar{B}'\right]x$$

or, $$\left(\frac{\sigma+\bar{\sigma}}{2}\right)\bar{x}'x = \bar{x}'\left[PD(\lambda)\bar{P}'RD(\mu)\bar{R}'\right]x$$
$$- \bar{x}'\left[QD(\gamma)\bar{Q}'SD(\nu)\bar{S}'\right]x. \qquad (11)$$

Now, by hypothesis, the Hermitian matrices $(A+\bar{A}')/2$, $(A-\bar{A}')/2i$, $(B+\bar{B}')/2$ and $(B-\bar{B}')/2i$ are at least positive semi-definite (to be called p.s.d.), so that $\lambda_i \geq 0$, $\gamma_i \geq 0$, $\mu_i \geq 0$ and $\nu_i \geq 0$ for $i = 1,2,\ldots,n$. Therefore the value of the first Hermitian form on the right-hand side of (11) at least does not decrease if we replace the diagonal matrices $D(\lambda)$ and $D(\mu)$, respectively, by the scalar matrices $\lambda_{max}I$ and $\mu_{max}I$, and the value of the second Hermitian form at least does not increase if we replace the diagonal matrices $D(\gamma)$ and $D(\nu)$, respectively, by the scalar matrices $\gamma_{min}I$ and $\nu_{min}I$. Thus we have

$$\left(\frac{\sigma+\bar{\sigma}}{2}\right)\bar{x}'x \leq \lambda_{max}\mu_{max}\left[\bar{x}'P\bar{P}'R\bar{R}'x\right] - \gamma_{min}\nu_{min}\left[\bar{x}'Q\bar{Q}'S\bar{S}'x\right]$$
$$= (\lambda_{max}\mu_{max} - \gamma_{min}\nu_{min})\bar{x}'x,$$

the matrices P, Q, R, and S being unitary.

Since $\bar{x}'x > 0$, we have

$$\frac{\sigma+\bar{\sigma}}{2} \leq \lambda_{max}\mu_{max} - \gamma_{min}\nu_{min}$$

or, $$\frac{c(AB) + \bar{c}(AB)}{2} \leq$$
$$c_{max}\left(\frac{A+\bar{A}'}{2}\right)c_{max}\left(\frac{B+\bar{B}'}{2}\right) - c_{min}\left(\frac{A-\bar{A}'}{2i}\right)c_{min}\left(\frac{B-\bar{B}'}{2i}\right). \qquad (12)$$

In order to find the lower bound for $(\sigma+\bar{\sigma})/2$ we observe that the value of the first Hermitian form on the right-hand side of (11) does not increase if we replace the diagonal matrices $D(\lambda)$ and $D(\mu)$, respectively, by the scalar matrices $\lambda_{min}I$ and $\mu_{min}I$, and the value of the second Hermitian form on the right-hand side of (11) does not decrease if we replace the diagonal matrices $D(\gamma)$ and $D(\vartheta)$, respectively, by the scalar matrices $\gamma_{max}I$ and $\vartheta_{max}I$. Thus we have

$$\left(\frac{\sigma+\bar{\sigma}}{2}\right)\bar{x}'x \geq \lambda_{min}\mu_{min}\left[\bar{x}'P\bar{P}'R\bar{R}'x\right] - \gamma_{max}\vartheta_{max}\left[\bar{x}'Q\bar{Q}'S\bar{S}'x\right]$$

$$= (\lambda_{min}\mu_{min} - \gamma_{max}\vartheta_{max})\bar{x}'x.$$

Since $\bar{x}'x > 0$, we have

$$\frac{\sigma+\bar{\sigma}}{2} \geq \lambda_{min}\mu_{min} - \gamma_{max}\vartheta_{max},$$

or, $$\frac{c(AB)+\bar{c}(AB)}{2} \geq$$

$$c_{min}\left(\frac{A+\bar{A}'}{2}\right)c_{min}\left(\frac{B+\bar{B}'}{2}\right) - c_{max}\left(\frac{A-\bar{A}'}{2i}\right)c_{max}\left(\frac{B-\bar{B}'}{2i}\right). \qquad (13)$$

Combining (12) and (13), we obtain (1).

In order to prove (2), we subtract (10) from (9), and have

$$(\sigma-\bar{\sigma})\bar{x}'x = \bar{x}'\left[AB - \bar{A}'\bar{B}'\right]x$$

$$= 2i\bar{x}'\left[PD(\lambda)\bar{P}'SD(\vartheta)\bar{S}'\right]x$$

$$+2i\bar{x}'\left[QD(\gamma)\bar{Q}'RD(\mu)\bar{R}'\right]x,$$

or, $$\left(\frac{\sigma-\bar{\sigma}}{2i}\right)\bar{x}'x = \bar{x}'\left[PD(\lambda)\bar{P}'SD(\vartheta)\bar{S}' + QD(\gamma)\bar{Q}'RD(\mu)\bar{R}'\right]x. \qquad (14)$$

By hypothesis, λ_i, γ_i, μ_i and ϑ_i for $i = 1,2,\ldots,n$ are non-negative. Therefore the value of the Hermitian form on the right-hand side of (14) does not decrease if we replace the diagonal matrices $D(\lambda)$, $D(\mu)$, $D(\gamma)$ and $D(\vartheta)$, respectively, by the scalar matrices $\lambda_{max}I$, $\mu_{max}I$, $\gamma_{max}I$ and $\vartheta_{max}I$, and we have

$$\frac{\sigma-\bar{\sigma}}{2i} \leq \lambda_{max}\vartheta_{max} + \gamma_{max}\mu_{max}, \text{ since } \bar{x}'x > 0,$$

or, $\dfrac{c(AB) - \bar{c}(AB)}{2i}$

$$\leq c_{max}\left(\frac{A+\bar{A}'}{2}\right)c_{max}\left(\frac{B-\bar{B}'}{2i}\right) + c_{max}\left(\frac{A-\bar{A}'}{2i}\right)c_{max}\left(\frac{B+\bar{B}'}{2}\right). \qquad (15)$$

Similarly, since $(A+\bar{A}')/2$, $(A-\bar{A}')/2i$, $(B+\bar{B}')/2$ and $(B-\bar{B}')/2i$ are at least p.s.d., λ_{min}, γ_{min}, μ_{min} and δ_{min} are non-negative, and we have

$$\frac{c(AB) - \bar{c}(AB)}{2i}$$

$$\geq c_{min}\left(\frac{A+\bar{A}'}{2}\right)c_{min}\left(\frac{B-\bar{B}'}{2i}\right) + c_{min}\left(\frac{A-\bar{A}'}{2i}\right)c_{min}\left(\frac{B+\bar{B}'}{2}\right). \qquad (16)$$

Combining (15) and (16) we establish (2) and this completes the proof of the theorem.

If the matrices A and B of Theorem 1 are such that the Hermitian matrices $(A+\bar{A}')/2$, $(A-\bar{A}')/2i$, $(B+\bar{B}')/2$ and $(B-\bar{B}')/2i$ are at least negative semi-definite (to be called n.s.d.), their characteristic roots are ≤ 0, so that in this case the above theorem takes the following form:

<u>Theorem</u> 2. <u>Let A and B be two commuting n-square complex matrices, such that the Hermitian matrices</u> $(A+\bar{A}')/2$, $(A-\bar{A}')/2i$, $(B+\bar{B}')/2$ <u>and</u> $(B-\bar{B}')/2i$ <u>are at least negative semi-definite.</u> <u>Then</u>

$$c_{max}\left(\frac{A+\bar{A}'}{2}\right)c_{max}\left(\frac{B+\bar{B}'}{2}\right) - c_{min}\left(\frac{A-\bar{A}'}{2i}\right)c_{min}\left(\frac{B-\bar{B}'}{2i}\right)$$

$$\leq \frac{c(AB) + \bar{c}(AB)}{2} \qquad (17)$$

$$\leq c_{min}\left(\frac{A+\bar{A}'}{2}\right)c_{min}\left(\frac{B+\bar{B}'}{2}\right) - c_{max}\left(\frac{A-\bar{A}'}{2i}\right)c_{max}\left(\frac{B-\bar{B}'}{2i}\right);$$

<u>and</u> $c_{max}\left(\frac{A+\bar{A}'}{2}\right)c_{max}\left(\frac{B-\bar{B}'}{2i}\right) + c_{max}\left(\frac{A-\bar{A}'}{2i}\right)c_{max}\left(\frac{B+\bar{B}'}{2}\right)$

$$\leq \frac{c(AB) - \bar{c}(AB)}{2i} \tag{18}$$

$$\leq c_{min}\left(\frac{A+\bar{A}'}{2}\right)c_{min}\left(\frac{B-\bar{B}'}{2i}\right) + c_{min}\left(\frac{A-\bar{A}'}{2i}\right)c_{min}\left(\frac{B+\bar{B}'}{2}\right),$$

where c_{max} and c_{min} denote, respectively, the greatest and and least characteristic roots.

We omit the proof of this theorem and consider the following:

Theorem 3. Let A and B be two commuting n-square complex matrices, such that $(A+\bar{A}')/2$ and $(A-\bar{A}')/2i$ are at least positive semi-definite, while $(B+\bar{B}')/2$ and $(B-\bar{B}')/2i$ are at least negative semi-definite. Then

$$c_{max}\left(\frac{A+\bar{A}'}{2}\right)c_{min}\left(\frac{B+\bar{B}'}{2}\right) - c_{min}\left(\frac{A-\bar{A}'}{2i}\right)c_{max}\left(\frac{B-\bar{B}'}{2i}\right)$$

$$\leq \frac{c(AB) + \bar{c}(AB)}{2} \tag{19}$$

$$\leq c_{min}\left(\frac{A+\bar{A}'}{2}\right)c_{max}\left(\frac{B+\bar{B}'}{2}\right) - c_{max}\left(\frac{A-\bar{A}'}{2i}\right)c_{min}\left(\frac{B-\bar{B}'}{2i}\right);$$

and

$$c_{max}\left(\frac{A+\bar{A}'}{2}\right)c_{min}\left(\frac{B-\bar{B}'}{2i}\right) + c_{max}\left(\frac{A-\bar{A}'}{2i}\right)c_{min}\left(\frac{B+\bar{B}'}{2}\right)$$

$$\leq \frac{c(AB) - \bar{c}(AB)}{2i} \tag{20}$$

$$\leq c_{min}\left(\frac{A+\bar{A}'}{2}\right)c_{max}\left(\frac{B-\bar{B}'}{2i}\right) + c_{min}\left(\frac{A-\bar{A}'}{2i}\right)c_{max}\left(\frac{B+\bar{B}'}{2}\right),$$

where c_{max} and c_{min} denote, respectively, the greatest and least characteristic roots.

Proof. Since the Hermitian matrices $(A+\bar{A}')/2$ and $(A-\bar{A}')/2i$ are at least p.s.d., $\lambda_i \geq 0$ and $\gamma_i \geq 0$ for $i=1,2,\ldots,n$; and since $(B+\bar{B}')/2$ and $(B-\bar{B}')/2i$ are at least n.s.d., $\mu_i \leq 0$ and $\nu_i \leq 0$ for $i=1,2,\ldots,n$. Now, if we replace the diagonal matrices $D(\lambda)$, $D(\mu)$, $D(\gamma)$ and $D(\nu)$ in (11), respectively, by the diagonal matrices $\lambda_{min}I$, $\mu_{max}I$, $\gamma_{max}I$ and $\delta_{min}I$ *the value of the Hermitian form on the right-hand side does* not decrease. Hence, we have

$$\frac{\sigma + \bar{\sigma}}{2} \leq \lambda_{min}\mu_{max} - \gamma_{max}\delta_{min},$$

or,
$$\frac{c(AB) + \bar{c}(AB)}{2} \leq$$
$$c_{min}\left(\frac{A+\bar{A}'}{2}\right)c_{max}\left(\frac{B+\bar{B}'}{2}\right) - c_{max}\left(\frac{A-\bar{A}'}{2i}\right)c_{min}\left(\frac{B-\bar{B}'}{2i}\right). \qquad (21)$$

Similarly, replacing the diagonal matrices $D(\lambda)$, $D(\mu)$, $D(\gamma)$ and $D(\delta)$ in (11) by the scalar matrices $\lambda_{max}I$, $\mu_{min}I$, $\gamma_{min}I$ and $\delta_{max}I$ we can prove that

$$\frac{\sigma + \bar{\sigma}}{2} \geq \lambda_{max}\mu_{min} - \gamma_{min}\delta_{max},$$

or,
$$\frac{c(AB) + \bar{c}(AB)}{2}$$
$$\geq c_{max}\left(\frac{A+\bar{A}'}{2}\right)c_{min}\left(\frac{B+\bar{B}'}{2}\right) - c_{min}\left(\frac{A-\bar{A}'}{2i}\right)c_{max}\left(\frac{B-\bar{B}'}{2i}\right). \qquad (22)$$

Combining the ineaqualities (21) and (22), we establish (19). Remembering the facts about the signs of the characteristic roots of the Hermitian matrices stated in the beginning of the proof of this theorem and replacing the diagonal matrices in (14) by suitable scalar matrices, we can likewise establish (20).

This completes the proof of Theorem 3.

So far we have considered only those matrices A and B for which the two pairs of the associated Hermitian matrices $(A+\bar{A}')/2$, $(A-\bar{A}')/2i$ and $(B+\bar{B}')/2$, $(B-\bar{B}')/2i$ are p.s.d. or n.s.d., or one pair is p.s.d. the other being n.s.d. We now consider the following theorem in which one pair of the associated Hermitian matrices is indefinite, while the other is at least p.s.d.:

Theorem 4. *Let A and B be two commuting n-square complex matrices, such that* $(A+\bar{A}')/2$ *and* $(A-\bar{A}')/2i$ *are indefinite, and* $(B+\bar{B}')/2$ *and* $(B-\bar{B}')/2i$ *are at least positive semi-definite. Then*

$$c_{min}\left(\frac{A+\bar{A}'}{2}\right)c_{max}\left(\frac{B+\bar{B}'}{2}\right) - c_{max}\left(\frac{A-\bar{A}'}{2i}\right)c_{max}\left(\frac{B-\bar{B}'}{2i}\right)$$

$$\leq \frac{c(AB) + \bar{c}(AB)}{2} \tag{23}$$

$$\leq c_{max}\left(\frac{A+\bar{A}'}{2}\right)c_{max}\left(\frac{B+\bar{B}'}{2}\right) - c_{min}\left(\frac{A-\bar{A}'}{2i}\right)c_{max}\left(\frac{B-\bar{B}'}{2i}\right);$$

and

$$c_{min}\left(\frac{A+\bar{A}'}{2}\right)c_{max}\left(\frac{B-\bar{B}'}{2i}\right) + c_{min}\left(\frac{A-\bar{A}'}{2i}\right)c_{max}\left(\frac{B+\bar{B}'}{2}\right)$$

$$\leq \frac{c(AB) - \bar{c}(AB)}{2i} \tag{24}$$

$$\leq c_{max}\left(\frac{A+\bar{A}'}{2}\right)c_{max}\left(\frac{B-\bar{B}'}{2i}\right) + c_{max}\left(\frac{A-\bar{A}'}{2i}\right)c_{max}\left(\frac{B+\bar{B}'}{2}\right),$$

where c_{max} *and* c_{min} *denote, respectively, the greatest and least characteristic roots.*

The inequalities of this theorem can be established by replacing the diagonal matrices of (11) and (14) by suitable scalar matrices taking into account the facts that since $(A+\bar{A}')/2$ and $(A-\bar{A}')/2i$ are indefinite $\lambda_{max} > 0$, $\lambda_{min} < 0$ $\gamma_{max} > 0$ and $\gamma_{min} < 0$ and since $(B+\bar{B}')/2$ and $(B-\bar{B}')/2i$ are at least p.s.d., their characteristic roots are non-negative.

As we shall see in section 3, the four theorems so far established, are such that for particular values of B, they give limits to the real and imaginary parts of characteristic roots of A which is such that the two matrices $(A+\bar{A}')/2$ and $(A-\bar{A}')/2i$ are (i) at least p.s.d., (ii) at least n.s.d., or (iii) indefinite. But the theorems of Hirsch and Bromwich, giving limits to the real and imaginary parts of $c(A)$ in terms of the characteristic roots of $(A+\bar{A}')/2$ and $(A-\bar{A}')/2i$, respectively, are true even if A is such that

EITHER (i). one of the two matrices $(A+\bar{A}')/2$ and $(A-\bar{A}')/2i$ is at least p.s.d., the other being indefinite,

OR (ii). one of the two matrices $(A+\bar{A}')/2$ and $(A-\bar{A}')/2i$ is at least p.s.d., the other being at least n.s.d.,

OR (iii). one of the two matrices $(A+\bar{A}')/2$ and $(A-\bar{A}')/2i$ is indefinite, the other being at least n.s.d.

The above-mentioned cases of theorems of Hirsch and Bromwich, not contained in Theorems 1,2,3,4, are contained in the following theorem which has been simply stated here and can be proved by following arguments parallel to those used in the theorems already proved:

<u>Theorems</u> 5. <u>Let A and B be two commuting n-square complex matrices, such that the Hermitian matrices</u> $(B+\bar{B}')/2$ <u>and</u> $(B-\bar{B}')/2i$ <u>are at least p.s.d. Then</u>

(a). <u>If</u> $(A+\bar{A}')/2$ <u>is indefinite and</u> $(A-\bar{A}')/2i$ <u>is at least positive semi-definite</u>,

$$c_{min}\left(\frac{A+\bar{A}'}{2}\right)c_{max}\left(\frac{B+\bar{B}'}{2}\right) - c_{max}\left(\frac{A-\bar{A}'}{2i}\right)c_{max}\left(\frac{B-\bar{B}'}{2i}\right)$$

$$\leq \frac{c(AB) + \bar{c}(AB)}{2} \tag{25}$$

$$\leq c_{max}\left(\frac{A+\bar{A}'}{2}\right)c_{max}\left(\frac{B+\bar{B}'}{2}\right) - c_{min}\left(\frac{A-\bar{A}'}{2i}\right)c_{min}\left(\frac{B-\bar{B}'}{2i}\right);$$

and

$$c_{min}\left(\frac{A+\bar{A}'}{2}\right)c_{max}\left(\frac{B-\bar{B}'}{2i}\right) + c_{min}\left(\frac{A-\bar{A}'}{2i}\right)c_{min}\left(\frac{B+\bar{B}'}{2}\right)$$

$$\leq \frac{c(AB) - \bar{c}(AB)}{2i} \tag{26}$$

$$\leq c_{max}\left(\frac{A+\bar{A}'}{2}\right)c_{max}\left(\frac{B-\bar{B}'}{2i}\right) + c_{max}\left(\frac{A-\bar{A}'}{2i}\right)c_{max}\left(\frac{B+\bar{B}'}{2}\right);$$

(b). *if* $(A+\bar{A}')/2$ *is at least p.s.d. and* $(A-\bar{A}')/2i$ *is indefinite*,

$$c_{min}\left(\frac{A+\bar{A}'}{2}\right)c_{min}\left(\frac{B+\bar{B}'}{2}\right) - c_{max}\left(\frac{A-\bar{A}'}{2i}\right)c_{max}\left(\frac{B-\bar{B}'}{2i}\right)$$

$$\leq \frac{c(AB) + \bar{c}(AB)}{2} \tag{27}$$

$$\leq c_{max}\left(\frac{A+\bar{A}'}{2}\right)c_{max}\left(\frac{B+\bar{B}'}{2}\right) - c_{min}\left(\frac{A-\bar{A}'}{2i}\right)c_{max}\left(\frac{B-\bar{B}'}{2i}\right),$$

and

$$c_{min}\left(\frac{A+\bar{A}'}{2}\right)c_{min}\left(\frac{B-\bar{B}'}{2i}\right) + c_{min}\left(\frac{A-\bar{A}'}{2i}\right)c_{max}\left(\frac{B+\bar{B}'}{2}\right)$$

$$\leq \frac{c(AB) - \bar{c}(AB)}{2i} \tag{28}$$

$$\leq c_{max}\left(\frac{A+\bar{A}'}{2}\right)c_{max}\left(\frac{B-\bar{B}'}{2i}\right) + c_{max}\left(\frac{A-\bar{A}'}{2i}\right)c_{max}\left(\frac{B+\bar{B}'}{2}\right);$$

(c). *if* $(A+\bar{A}')/2$ *is at least n.s.d. and* $(A-\bar{A}')/2i$ *is at least p.s.d.*,

$$c_{min}\left(\frac{A+\bar{A}'}{2}\right)c_{max}\left(\frac{B+\bar{B}'}{2}\right) - c_{max}\left(\frac{A-\bar{A}'}{2i}\right)c_{max}\left(\frac{B-\bar{B}'}{2i}\right)$$

$$\leq \frac{c(AB) + \bar{c}(AB)}{2} \tag{29}$$

$$\leq c_{max}\left(\frac{A+\bar{A}'}{2}\right)c_{min}\left(\frac{B+\bar{B}'}{2}\right) - c_{min}\left(\frac{A-\bar{A}'}{2i}\right)c_{min}\left(\frac{B-\bar{B}'}{2i}\right),$$

and

$$c_{min}\left(\frac{A+\bar{A}'}{2}\right)c_{max}\left(\frac{B-\bar{B}'}{2i}\right) + c_{min}\left(\frac{A-\bar{A}'}{2i}\right)c_{min}\left(\frac{B+\bar{B}'}{2}\right)$$

$$\leq \quad \frac{c(AB) - \bar{c}(AB)}{2i} \tag{30}$$

$$\leq c_{max}\left(\frac{A+\bar{A}'}{2}\right)c_{min}\left(\frac{B-\bar{B}'}{2i}\right) + c_{max}\left(\frac{A-\bar{A}'}{2i}\right)c_{max}\left(\frac{B+\bar{B}'}{2}\right);$$

(d). <u>if</u> $(A+\bar{A}')/2$ <u>is at least p.s.d. and</u> $(A-\bar{A}')/2i$ <u>is at least n.s.d.,</u>

$$c_{min}\left(\frac{A+\bar{A}'}{2}\right)c_{min}\left(\frac{B+\bar{B}'}{2}\right) - c_{max}\left(\frac{A-\bar{A}'}{2i}\right)c_{min}\left(\frac{B-\bar{B}'}{2i}\right)$$

$$\leq \quad \frac{c(AB) + \bar{c}(AB)}{2} \tag{31}$$

$$\leq c_{max}\left(\frac{A+\bar{A}'}{2}\right)c_{max}\left(\frac{B+\bar{B}'}{2}\right) - c_{min}\left(\frac{A-\bar{A}'}{2i}\right)c_{max}\left(\frac{B-\bar{B}'}{2i}\right),$$

<u>and</u>

$$c_{min}\left(\frac{A+\bar{A}'}{2}\right)c_{min}\left(\frac{B-\bar{B}'}{2i}\right) + c_{min}\left(\frac{A-\bar{A}'}{2i}\right)c_{max}\left(\frac{B+\bar{B}'}{2}\right)$$

$$\leq \quad \frac{c(AB) - \bar{c}(AB)}{2i} \tag{32}$$

$$\leq c_{max}\left(\frac{A+\bar{A}'}{2}\right)c_{max}\left(\frac{B-\bar{B}'}{2i}\right) + c_{max}\left(\frac{A-\bar{A}'}{2i}\right)c_{min}\left(\frac{B+\bar{B}'}{2}\right);$$

(e). <u>if</u> $(A+\bar{A}')/2$ <u>is indefinite and</u> $(A-\bar{A}')/2i$ <u>is at least n.s.d.,</u>

$$c_{min}\left(\frac{A+\bar{A}'}{2}\right)c_{max}\left(\frac{B+\bar{B}'}{2}\right) - c_{max}\left(\frac{A-\bar{A}'}{2i}\right)c_{min}\left(\frac{B-\bar{B}'}{2i}\right)$$

$$\leq \quad \frac{c(AB) + \bar{c}(AB)}{2} \tag{33}$$

$$\leq c_{max}\left(\frac{A+\bar{A}'}{2}\right)c_{max}\left(\frac{B+\bar{B}'}{2}\right) - c_{min}\left(\frac{A-\bar{A}'}{2i}\right)c_{max}\left(\frac{B-\bar{B}'}{2i}\right),$$

<u>and</u>

$$c_{min}\left(\frac{A+\bar{A}'}{2}\right)c_{max}\left(\frac{B-\bar{B}'}{2i}\right) + c_{min}\left(\frac{A-\bar{A}'}{2i}\right)c_{max}\left(\frac{B+\bar{B}'}{2}\right)$$

$$\leq \quad \frac{c(AB) - \bar{c}(AB)}{2i} \tag{34}$$

$$\leq c_{max}\left(\frac{A+\bar{A}'}{2}\right)c_{max}\left(\frac{B-\bar{B}'}{2i}\right) + c_{max}\left(\frac{A-\bar{A}'}{2i}\right)c_{min}\left(\frac{B+\bar{B}'}{2}\right);$$

(f). <u>if</u> $(A+\bar{A}')/2$ <u>is at least n.s.d. and</u> $(A-\bar{A}')/2i$ <u>is</u>

indefinite,

$$c_{min}\left(\frac{A+\bar{A}'}{2}\right)c_{max}\left(\frac{B+\bar{B}'}{2}\right) - c_{max}\left(\frac{B-\bar{B}'}{2i}\right)c_{max}\left(\frac{A-\bar{A}'}{2i}\right)$$

$$\leq \frac{c(AB) + \bar{c}(AB)}{2} \tag{35}$$

$$\leq c_{max}\left(\frac{A+\bar{A}'}{2}\right)c_{min}\left(\frac{B+\bar{B}'}{2}\right) - c_{min}\left(\frac{A-\bar{A}'}{2i}\right)c_{max}\left(\frac{B-\bar{B}'}{2i}\right),$$

and

$$c_{min}\left(\frac{A+\bar{A}'}{2}\right)c_{max}\left(\frac{B-\bar{B}'}{2i}\right) + c_{min}\left(\frac{A-\bar{A}'}{2i}\right)c_{max}\left(\frac{B+\bar{B}'}{2}\right)$$

$$\leq \frac{c(AB) - \bar{c}(AB)}{2i} \tag{36}$$

$$\leq c_{max}\left(\frac{A+\bar{A}'}{2}\right)c_{min}\left(\frac{B-\bar{B}'}{2i}\right) + c_{max}\left(\frac{A-\bar{A}'}{2i}\right)c_{max}\left(\frac{B+\bar{B}'}{2}\right),$$

where c_{max} and c_{min} denote, respectively, the greatest and least characteristic roots.

3. Some special cases of the above theorems.

We now give certain examples to show that many interesting results can be obtained as particular cases of the above theorems.

(i). If we put $B = I$ in Theorems 1 and 4, for which $(B+\bar{B}')/2 = I$ and $B-\bar{B}' = 0$, we have

$$c_{min}\left(\frac{A+\bar{A}'}{2}\right) \leq \frac{c(A) + \bar{c}(A)}{2} \leq c_{max}\left(\frac{A+\bar{A}'}{2}\right), \tag{37}$$

and $$c_{min}\left(\frac{A-\bar{A}'}{2i}\right) \leq \frac{c(A) - \bar{c}(A)}{2i} \leq c_{max}\left(\frac{A-\bar{A}'}{2i}\right), \tag{38}$$

results due to Hirsch [2] and Bromwich [1] , (when $(A+\bar{A}')/2$ and $(A-\bar{A}')/2i$ are at least p.s.d. or indefinite).

It may be noted that (37) and (38) can also be derived from Theorems 1 and 4 by putting $B=iI$.

(ii). If the the two Hermitian matrices A and B commute and are at least p.s.d., then AB is also Hermitian and at least p.s.d. so that all $c(AB)$'s are real and nonnegative. The limits for $c(AB)$, will, then, be given by (1) which reduces to

$$c_{min}(A)c_{min}(B) \leq c(AB) \leq c_{max}(A)c_{max}(B). \quad (39)$$

(iii). If A is skew-Hermitian, then $A+\bar{A}' = 0$. If, moreover, A is such that the Hermitian matrix $(A-\bar{A}')/2i = A/i = -iA$ is at least p.s.d., then every $c(-iA)$ is real and ≥ 0, so that $c(iA) \leq 0$. Also, $c_{min}(-iA) = -c_{max}(iA)$, $c_{max}(-iA) = -c_{min}(iA)$. Similar relations hold for B if it is skew-Hermitian and $-iB$ is at least p.s.d. The product $-AB$ of the two at least p.s.d. Hermitian matrices A/i and B/i is also Hermitian and p.s.d., so that $c(-AB)$'s are real and ≥ 0 and $c(AB) \leq 0$. For the commuting skew-Hermitian matrices A and B defined above, (1) reduces to

$$-c_{max}(-iA)c_{max}(-iB) \leq c(AB) \leq -c_{min}(-iA)c_{min}(-iB)$$

or, $$-c_{min}(iA)c_{min}(iB) \leq c(AB) \leq -c_{max}(iA)c_{max}(iB)$$

or, $$c_{max}(iA)c_{max}(iB) \leq c(-AB) \leq c_{min}(iA)c_{min}(iB). \quad (40)$$

(iv). If A and B are skew-Hermitian matrices such that they commute and $-iA$ and $-iB$ are at least n.s.d., then arguing as we did in establishing (40), the following can be derived from (17):

$$c_{min}(iA)c_{min}(iB) \leq c(-AB) \leq c_{max}(iA)c_{max}(iB). \quad (41)$$

(v). If we put $B = -iI$ for which $B+\bar{B}' = 0$, $(B-\bar{B}')/2i = -I$, a negative definite Hermitian matrix, (17) reduces to

$$c_{min}\left(\frac{A-\bar{A}'}{2i}\right) \leq \frac{c(-iA) + \bar{c}(-iA)}{2} \leq c_{max}\left(\frac{A-\bar{A}'}{2i}\right)$$

or, $$c_{min}\left(\frac{A-\bar{A}'}{2i}\right) \leq \frac{-ic(A) + i\bar{c}(A)}{2} \leq c_{max}\left(\frac{A-\bar{A}'}{2i}\right)$$

or, $c_{min}(\frac{A-\bar{A}'}{2i}) \leq \frac{c(A) - \bar{c}(A)}{2i} \leq c_{max}(\frac{A-\bar{A}'}{2i})$, (37')

due to Bromwich [1] , (when $(A-\bar{A}')/2i$ is at least n.s.d.).

The same substitution reduces (18) to

$$c_{min}(\frac{A+\bar{A}'}{2}) \leq \frac{c(A) + \bar{c}(A)}{2} \leq c_{max}(\frac{A+\bar{A}'}{2}), \qquad (38')$$

a result due to Hirsch [2], (when $(A+\bar{A}')/2$ is at least n.s.d.).

(vi). If we put $B = I$ in the results of Theorem 5, they reduce to

$$c_{min}(\frac{A+\bar{A}'}{2}) \leq \frac{c(A) + \bar{c}(A)}{2} \leq c_{max}(\frac{A+\bar{A}'}{2}), \qquad (37'')$$

and $c_{min}(\frac{A-\bar{A}'}{2i}) \leq \frac{c(A) - \bar{c}(A)}{2i} \leq c_{max}(\frac{A-\bar{A}'}{2i})$, (38'')

when A is such that

__Either__ (i) one of the two matrices $(A+\bar{A}')/2$ and $(A-\bar{A}')/2i$ is at least pos.d., the other being indefinite;

Or, (ii) one of the two matrices $(A+\bar{A}')/2$ and $(A-\bar{A}')/2i$ is at least p.s.d., the other being at least n.s.d.;

Or, (iii) one of the two matrices $(A+\bar{A}')/2$ and $(A-\bar{A}')/2i$ is at least n.s.d., the other being indefinite.

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THE CHARACTERISTIC ROOTS OF THE PRODUCT OF MATRICES V.

1. Introduction.

In this paper we shall find the upper and lower limits for the real and imaginary parts of the characteristic roots of the product matrix, AB, of two n-square real matrices A and B in terms of the characteristic roots of the auxiliary matrices $(A+A')/2$, $(A-A')/2i$, $(B+B')/2$ and $(B-B')/2i$. The results established here will contain the following results of Bendixon [1] and Bromwich [2] as particular cases:

Bendixon's theorem. If $\alpha+i\beta$ is a characteristic root of a real n-square matrix A and if $\rho_1 \leq \rho_2 \leq \dots \leq \rho_n$ are the (necessarily real) characteristic roots of $(A+A')/2$, then

$$\rho_1 \leq \alpha \leq \rho_n .$$

Bromwich's theorem. If $\alpha+i\beta$ is a characteristic root of a real n-square matrix A and if we denote by $\pm i\mu_1$, $\pm i\mu_2, \dots,$ $\pm i\mu_r$, $2r \leq n$, the nonzero characteristic roots of the matrix $(A-A')/2$, then

$$|\beta| \leq \max |\mu_i|.$$

As usual, $c(AB)$ denotes an arbitrary characteristic root of AB and $\bar{c}(AB)$, the complex conjugate of $c(AB)$.

2. Limits for the real parts of $c(AB)$.

Theorem 1. Let A and B be two commuting n-square real matrices such that the two symmetric matrices $(A+A')/2$ and $(B+B')/2$ are

<u>at least positive semi-definite. Then</u>

$$c_{min}\left(\frac{A+A'}{2}\right)c_{min}\left(\frac{B+B'}{2}\right) - c_{max}\left(\frac{A-A'}{2i}\right)c_{max}\left(\frac{B-B'}{2i}\right)$$

$$\leq \frac{c(AB) + \bar{c}(AB)}{2} \tag{1}$$

$$\leq c_{max}\left(\frac{A+A'}{2}\right)c_{max}\left(\frac{B+B'}{2}\right) - c_{min}\left(\frac{A-A'}{2i}\right)c_{max}\left(\frac{B-B'}{2i}\right),$$

<u>where c_{max} and c_{min} denote, respectively, the greatest and least characteristic roots.</u>

<u>Proof</u>. Every real square matrix $A = (A+A')/2 + (A-A')/2$, where $(A+A')/2$ is symmetric and $(A-A')/2$ is skew-symmetric. Let the (necessarily real but not necessarily distinct) characteristic roots of $(A+A')/2$ be $\lambda_1, \lambda_2, \ldots, \lambda_n$, arranged in any desired order. Then there exists an orthogonal matrix P, such that

$$P'\left(\frac{A+A'}{2}\right)P = \text{diag.}(\lambda_1, \lambda_2, \ldots, \lambda_n) \equiv D(\lambda), \text{ say.}$$

Therefore

$$\frac{A+A'}{2} = PD(\lambda)P'. \tag{2}$$

The matrix $(A-A')/2$ is skew-symmetric. Its nonzero characteristic roots are of the form $\pm i\alpha_1, \pm i\alpha_2, \ldots, \pm i\alpha_r$, where α's are all real and may be supposed to be positive without any loss of generality and $2r \leq n$, so that there are $n-2r$ zero roots of $(A-A')/2$. We can further suppose that the values of $c\left(\frac{A-A'}{2}\right)$ are $i\gamma_1, i\gamma_2, \ldots, i\gamma_n$. (Here some γ's may be zero while the rest shall be divided into pairs of real numbers which are equal in magnitude but opposite in sign). It may be seen that $(A-A')/2i$ is Hermitian and its characteristic roots are $\gamma_1, \gamma_2, \ldots, \gamma_n$, so that there exists a unitary matrix Q, such that

$$\frac{A-A'}{2i} = QD(\gamma)\bar{Q}', \qquad (3)$$

where $D(\gamma) \equiv \text{diag.}(\gamma_1, \gamma_2, \ldots, \gamma_n)$.

From (3), we have

$$\frac{A-A'}{2} = iQD(\gamma)\bar{Q}'. \qquad (4)$$

Thus

$$A = PD(\lambda)P' + iQD(\gamma)\bar{Q}'. \qquad (5)$$

Now, let $\mu_1, \mu_2, \ldots, \mu_n$ be the (real) characteristic roots of the symmetric matrix $(B+B')/2$ and let $\delta_1, \delta_2, \ldots, \delta_n$ be the (real) characteristic roots of the skew-Hermitian matrix $(B-B')/2i$. Here, some δ's may be zero while the rest can be divided into pairs of real numbers equal in magnitude but opposite in sign. Then proceeding as we did in the case of A, we have

$$B = RD(\mu)R' + iSD(\delta)\bar{S}', \qquad (6)$$

where $D(\mu) \equiv \text{diag.}(\mu_1, \mu_2, \ldots, \mu_n)$, $D(\delta) \equiv \text{diag.}(\delta_1, \delta_2, \ldots, \delta_n)$, R is an orthogonal matrix and S is unitary.

From (5) and (6), we have

$$AB = PD(\lambda)P'RD(\mu)R' - QD(\gamma)\bar{Q}'SD(\delta)\bar{S}' + i\left[PD(\lambda)P'SD(\delta)\bar{S}' + QD(\gamma)\bar{Q}'RD(\mu)R'\right]. \qquad (7)$$

If σ is a charcteristic root of AB, there exists a column vector $x = (x_1, x_2, \ldots, x_n)$ with complex coordinates not all equal to zero, such that

$$\sigma x = ABx. \qquad (8)$$

Premultiplying both the sides of the above equation by $\bar{x}'$, we have

$$\sigma\, \bar{x}'x \;=\; \bar{x}'ABx \tag{9}$$

Taking the conjugate transpose of the above equation, we have

$$\begin{aligned}\bar{\sigma}\, \bar{x}'x &= \bar{x}'(B'A')x \\ &= \bar{x}'(A'B')x\,,\end{aligned} \tag{10}$$

since $AB = BA$ implies that $B'A' = A'B'$, where

$$\begin{aligned} A'B' = \bar{A}\bar{B}' &= PD(\lambda)P'RD(\mu)R' - QD(\gamma)\bar{Q}'SD(\delta)\bar{S}' \\ &\quad -i\left[PD(\lambda)P'SD(\delta)\bar{S}' + QD(\gamma)\bar{Q}'RD(\mu)R'\right]. \end{aligned} \tag{11}$$

Adding (9) and (10), side by side, we have

$$\begin{aligned}\left(\frac{\sigma+\bar{\sigma}}{2}\right)\bar{x}'x &= \bar{x}'\left[PD(\lambda)P'RD(\mu)R'\right]x \\ &\quad - \bar{x}'\left[QD(\gamma)\bar{Q}'SD(\delta)\bar{S}'\right]x\,. \end{aligned} \tag{12}$$

Now, the symmetric matrices $(A+A')/2$ and $(B+B')/2$ are at least positive semi-definite (to be called p.s.d.), so that λ's and μ's are all real nonnegative quantities. Therefore, if we replace the diagonal matrices $D(\lambda)$ and $D(\mu)$, respectively, by the scalar matrices $\lambda_{max}I$ and $\mu_{max}I$, the first term on the right-hand side of (12) increases (at least it does not decrease). Similarly, if we replace the diagonal matrices $D(\gamma)$ and $D(\delta)$, respectively, by the scalar matrices $\gamma_{min}I$ and $\delta_{max}I$, the second term on the right-hand side of (12), prefixed by minus sign, becomes negative and numerically greatest, for γ_{min} is negative and δ_{max} is positive, both of them being numerically greatest. Thus due to these replacements, the right-hand side of (12) increases (at least it does not decrease), and we have

$$\left(\frac{\sigma+\bar{\sigma}}{2}\right)\bar{x}'x \leq \lambda_{max}\mu_{max}(\bar{x}'PP'RR'x) - \gamma_{min}\delta_{max}(\bar{x}'Q\bar{Q}'S\bar{S}'x).$$

Since the matrices P and R are orthogonal $PP' = RR' = I$, and since Q and S are unitary, $Q\bar{Q}' = S\bar{S}' = I$, so that we have

$$\left(\frac{\sigma+\bar{\sigma}}{2}\right)\bar{x}'x \leq (\lambda_{max}\mu_{max} - \gamma_{min}\vartheta_{max})\bar{x}'x.$$

But, since $\bar{x}'x > 0$, it follows that

$$\frac{\sigma+\bar{\sigma}}{2} \leq \lambda_{max}\mu_{max} - \gamma_{min}\vartheta_{max}.$$

Thus

$$\frac{c(AB) + \bar{c}(AB)}{2}$$

$$\leq c_{max}\left(\frac{A+A'}{2}\right)c_{max}\left(\frac{B+B'}{2}\right) - c_{min}\left(\frac{A-A'}{2i}\right)c_{max}\left(\frac{B-B'}{2i}\right). \tag{13}$$

In order to find the lower limit for $\frac{\sigma+\bar{\sigma}}{2}$, let us replace the diagonal matrices $D(\lambda)$ and $D(\mu)$, respectively, by the scalar matrices $\lambda_{min}I$ and $\mu_{min}I$, where λ_{min} and μ_{min} are nonnegative quantities. The first term on the right-hand side of (12), thus, decreases (at least it does not increase). If, further, we replace the diagonal matrices $D(\gamma)$ and $D(\vartheta)$, respectively, by the scalar matrices $\gamma_{max}I$ and $\vartheta_{max}I$, the second term on the right-hand side of (12), prefixed by minus sign, at least does not decrease. Thus, due to these replacements the right-hand side of (12) decreases (at least it does not increase), and we have

$$\left(\frac{\sigma+\bar{\sigma}}{2}\right)\bar{x}'x \geq \lambda_{min}\mu_{min}(\bar{x}'PP'RR'x) - \gamma_{max}\vartheta_{max}(\bar{x}'Q\bar{Q}'S\bar{S}'x)$$

$$= (\lambda_{min}\mu_{min} - \gamma_{max}\vartheta_{max})\bar{x}'x,$$

since $PP' = RR' = Q\bar{Q}' = S\bar{S}' = I$.

But, since $\bar{x}'x > 0$, we have

$$\frac{\sigma+\bar{\sigma}}{2} \geq \lambda_{min}\mu_{min} - \gamma_{max}\vartheta_{max}.$$

or, $$\frac{c(AB) + \bar{c}(AB)}{2} \geq$$

$$c_{min}\left(\frac{A+A'}{2}\right)c_{min}\left(\frac{B+B'}{2}\right) - c_{max}\left(\frac{A-A'}{2i}\right)c_{max}\left(\frac{B-B'}{2i}\right). \qquad (14)$$

Combining (13) and (14), we obtain (1), and this completes the proof of Theorem 1.

We now consider the real parts of $c(AB)$ when one of the two matrices A and B, say B, is such that $(B+B')/2$ is at least p.s.d., while A is such that $(A+A')/2$ is at least negative semi-definite (to be called n.s.d.), and prove the following theorem:

<u>Theorem</u> 2. <u>Let A and B be two commuting n-square real matrices such that the symmetric matrix $(A+A')/2$ is at least negative semi-definite and $(B+B')/2$ is at least positive semi-definite. Then</u>

$$c_{min}\left(\frac{A+A'}{2}\right)c_{max}\left(\frac{B+B'}{2}\right) - c_{max}\left(\frac{A-A'}{2i}\right)c_{max}\left(\frac{B-B'}{2i}\right)$$

$$\leq \frac{c(AB) + \bar{c}(AB)}{2} \qquad (15)$$

$$\leq c_{max}\left(\frac{A+A'}{2}\right)c_{min}\left(\frac{B+B'}{2}\right) - c_{min}\left(\frac{A-A'}{2i}\right)c_{max}\left(\frac{B-B'}{2i}\right),$$

<u>where c_{max} and c_{min} stand, respectively, for the greatest and least charcteristic roots.</u>

<u>Proof</u>. Since the matrix $(B+B')/2$ is at least p.s.d., $\mu_i \geq 0$ for $i = 1, 2, \ldots, n$; and since $(A+A')/2$ is at least n.s.d., $\lambda_i \leq 0$ for $i = 1, 2, \ldots, n$. Now, if we replace the diagonal matrices $D(\lambda)$ and $D(\mu)$, respectively, by the scalar matrices $\lambda_{max}I$ and $\mu_{min}I$, the first term on the right- hand

side of (12) does not at least decrease. If we replace the diagonal matrices $D(\gamma)$ and $D(\delta)$, respectively, by the scalar matrices $\gamma_{min}I$ and $\delta_{max}I$, we see that the second term on the right-hand side of (12) does not at least increase. Hence we have

$$\frac{\sigma+\bar{\sigma}}{2} \leq \lambda_{max}\mu_{min} - \gamma_{min}\delta_{max},$$ since P and R are orthogonal, Q and S are unitary, and $\bar{x}'x > 0$.

or, $$\frac{c(AB)+\bar{c}(AB)}{2} \leq$$

$$c_{max}\left(\frac{A+A'}{2}\right)c_{min}\left(\frac{B+B'}{2}\right) - c_{min}\left(\frac{A-A'}{2i}\right)c_{max}\left(\frac{B-B'}{2i}\right). \qquad (16)$$

Similarly, replacing the diagonal matrices $D(\lambda)$, $D(\mu)$, $D(\gamma)$ and $D(\delta)$, on the right-hand side of (12), respectively, by the scalar matrices $\lambda_{min}I$, $\mu_{max}I$, $\gamma_{max}I$ and $\delta_{max}I$, we can prove that

$$\frac{c(AB)+\bar{c}(AB)}{2} \geq$$

$$c_{min}\left(\frac{A+A'}{2}\right)c_{max}\left(\frac{B+B'}{2}\right) - c_{max}\left(\frac{A-A'}{2i}\right)c_{max}\left(\frac{B-B'}{2i}\right). \qquad (17)$$

Combining (16) and (17), we establish (15).

We now consider below the real parts of c(AB), when (A+A')/2 is indefinite and (B+B')/2 is at least p.s.d. :

Theorem 3. Let A and B be two commuting n-square real matrices such that the symmetric matrix (A+A')/2 is indefinite and (B+B')/2 is at least positive semi-definite. Then

$$c_{min}\left(\frac{A+A'}{2}\right)c_{max}\left(\frac{B+B'}{2}\right) - c_{max}\left(\frac{A-A'}{2i}\right)c_{max}\left(\frac{B-B'}{2i}\right)$$

$$\leq \frac{c(AB) + \bar{c}(AB)}{2} \tag{18}$$

$$\leq c_{max}\left(\frac{A+A'}{2}\right)c_{max}\left(\frac{B+B'}{2}\right) - c_{min}\left(\frac{A-A'}{2i}\right)c_{max}\left(\frac{B-B'}{2i}\right),$$

where c_{max} and c_{min} stand, respectively, for the greatest and least characteristic roots.

Proof. Since the symmetric matrix $(B+B')/2$ is al least p.s.d., $\mu_i \geq 0$ for $i = 1,2,\ldots,n$; and since $(A+A')/2$ is indefinite, $\lambda_{max} > 0$ and $\lambda_{min} < 0$. Hence, replacing the diagonal matrices $D(\lambda)$, $D(\mu)$, $D(\gamma)$ and $D(\delta)$ on the right-hand side of (12), respectively, by the scalar matrices $\lambda_{max}I$, $\mu_{max}I$, $\gamma_{min}I$ and $\delta_{max}I$, and remembering that P, R are orthogonal, Q, S are unitary and $\bar{x}'x > 0$, we can prove that

$$\frac{c(AB) + \bar{c}(AB)}{2} \leq$$

$$c_{max}\left(\frac{A+A'}{2}\right)c_{max}\left(\frac{B+B'}{2}\right) - c_{min}\left(\frac{A-A'}{2i}\right)c_{max}\left(\frac{B-B'}{2i}\right). \tag{19}$$

Similarly, if we replace the diagonal matrices $D(\lambda)$, $D(\mu)$, $D(\gamma)$ and $D(\delta)$ on the right-hand side of (12), respectively, by the scalar matrices $\lambda_{min}I$, $\mu_{max}I$, $\gamma_{max}I$ and $\delta_{max}I$, we have

$$\frac{c(AB) + \bar{c}(AB)}{2} \geq$$

$$c_{min}\left(\frac{A+A'}{2}\right)c_{max}\left(\frac{B+B'}{2}\right) - c_{max}\left(\frac{A-A'}{2i}\right)c_{max}\left(\frac{B-B'}{2i}\right). \tag{20}$$

Combining (19) and (20), we obtain (18).

3. Limits for the imaginary parts of c(AB).

We shall, now, find limits for the imaginary parts of $c(AB)$. The three different cases, viz., when $(A+A')/2$ is at least p.s.d., at least n.s.d. or indefinite, have been considered separately. But $(B+B')/2$ has throughout been taken to be at least p.s.d.

<u>Theorem</u> 4. <u>Let A and B be two commuting n-square real matrices, such that the symmetric matrices</u> $(A+A')/2$ <u>and</u> $(B+B')/2$ <u>are at least positive semi-definite. Then</u>

$$c_{max}\left(\frac{A+A'}{2}\right)c_{min}\left(\frac{B-B'}{2i}\right) + c_{min}\left(\frac{A-A'}{2i}\right)c_{max}\left(\frac{B+B'}{2}\right)$$

$$\leq \frac{c(AB) - \bar{c}(AB)}{2i} \tag{21}$$

$$\leq c_{max}\left(\frac{A+A'}{2}\right)c_{max}\left(\frac{B-B'}{2i}\right) + c_{max}\left(\frac{A-A'}{2i}\right)c_{max}\left(\frac{B+B'}{2}\right),$$

<u>where c_{max} and c_{min} stand, respectively, for the greatest and least characteristic roots.</u>

<u>Proof</u>. We have already established equations (9) and (10) while proving Theorem 1. In order to find the limits for the imaginary parts of $c(AB)$, we subtract (10) from (9), and have

$$(\sigma - \bar{\sigma})\bar{x}'x = \bar{x}'(AB - A'B')x,$$

or, $$\left(\frac{\sigma - \bar{\sigma}}{2i}\right)\bar{x}'x = \bar{x}'\left[PD(\lambda)P'SD(\vartheta)\bar{S}' + QD(\gamma)\bar{Q}'RD(\mu)R'\right]x. \tag{22}$$

Since $(A+A')/2$ and $(B+B')/2$ are at least p.s.d., their characteristic roots are nonnegative. Therefore, if we replace the diagonal matrices $D(\lambda)$, $D(\vartheta)$, $D(\gamma)$ and $D(\mu)$, respectively, by the scalar matrices $\lambda_{max}I$, $\vartheta_{max}I$, $\gamma_{max}I$ and $\mu_{max}I$, the value of the real Hermitian form on the right-hand side of (22) increases (at least it does not decrease) and we have

$$\left(\frac{\sigma - \bar{\sigma}}{2i}\right)\bar{x}'x \leq \lambda_{max}\,\vartheta_{max}\,(\bar{x}'PP'S\bar{S}'x)$$

$$+ \quad \gamma_{max}\,\mu_{max}\,(\bar{x}'Q\bar{Q}'RR'x)$$

$$= (\lambda_{max}\,\vartheta_{max} + \gamma_{max}\,\mu_{max})\bar{x}'x.$$

But, since $\bar{x}'x > 0$, we have

$$\frac{\sigma - \bar{\sigma}}{2i} \leq \lambda_{max}\,\vartheta_{max} + \gamma_{max}\,\mu_{max}$$

or, $$\frac{c(AB) - \bar{c}(AB)}{2i} \leq$$

$$c_{max}\left(\frac{A+A'}{2}\right)c_{max}\left(\frac{B-B'}{2i}\right) + c_{max}\left(\frac{A-A'}{2i}\right)c_{max}\left(\frac{B+B'}{2}\right). \tag{23}$$

In order to find the lower limits for $(\sigma - \bar{\sigma})/2i$, let us replace the diagonal matrices $D(\lambda)$, $D(\vartheta)$, $D(\gamma)$ and $D(\mu)$ on the right-hand side of (22), respectively, by the scalar matrices $\lambda_{max}I$, $\vartheta_{min}I$, $\gamma_{min}I$ and $\mu_{max}I$. Now, ϑ_{min} and γ_{min} are nonpositive and λ_{max} and μ_{max} are nonnegative real quantities, so that by these replacements the value of the real Hermitian form on the right-hand side of (22) is decreased (at least it is not increased), and we have

$$\left(\frac{\sigma - \bar{\sigma}}{2i}\right)\bar{x}'x \geq \lambda_{max}\,\vartheta_{min}\,(\bar{x}'PP'S\bar{S}'x) +$$

$$\gamma_{min}\,\mu_{max}\,(\bar{x}'Q\bar{Q}'RR'x),$$

$$= (\lambda_{max}\,\vartheta_{min} + \gamma_{min}\,\mu_{max})\bar{x}'x,$$

or, $$\frac{c(AB) - \bar{c}(AB)}{2i} \geq$$

$$c_{max}\left(\frac{A+A'}{2}\right)c_{min}\left(\frac{B-B'}{2i}\right) + c_{min}\left(\frac{A-A'}{2i}\right)c_{max}\left(\frac{B+B'}{2}\right). \tag{24}$$

Combining (23) and (24), we obtain (21).

Theorem 5. Let A and B be two commuting n-square real matrices, such that the symmetric matrix $(A+A')/2$ is at least negative semi-definite and $(B+B')/2$ is at least positive semi-definite. Then

$$c_{min}\left(\frac{A+A'}{2}\right)c_{max}\left(\frac{B-B'}{2i}\right) + c_{min}\left(\frac{A-A'}{2i}\right)c_{max}\left(\frac{B+B'}{2}\right)$$

$$\leq \frac{c(AB) - \bar{c}(AB)}{2i} \tag{25}$$

$$\leq c_{min}\left(\frac{A+A'}{2}\right)c_{min}\left(\frac{B-B'}{2i}\right) + c_{max}\left(\frac{A-A'}{2i}\right)c_{max}\left(\frac{B+B'}{2}\right),$$

where c_{max} and c_{min} stand, respectively, for the greatest and least characteristic roots.

Proof. Since the matrix $(B+B')/2$ is at least p.s.d., $\mu_i \geq 0$ for $i = 1,2,\ldots,n$; and since $(A+A')/2$ is at least n.s.d., $\lambda_i \leq 0$ for $i = 1,2,\ldots,n$. Now, to find the upper limit for $(\sigma - \bar{\sigma})/2i$, let us replace the diagonal matrices $D(\lambda)$, $D(\vartheta)$, $D(\gamma)$ and $D(\mu)$ on the right-hand side of (22), respectively, by the scalar matrices $\lambda_{min}I$, $\vartheta_{min}I$, $\gamma_{max}I$ and $\mu_{max}I$, where λ_{min} and ϑ_{min} are negative and numerically greatest. Thus the value of the real Hermitian form on the right-hand side of (22) at least does not decrease, and we have

$$\frac{c(AB) - \bar{c}(AB)}{2i} \leq$$

$$c_{min}\left(\frac{A+A'}{2}\right)c_{min}\left(\frac{B-B'}{2i}\right) + c_{max}\left(\frac{A-A'}{2i}\right)c_{max}\left(\frac{B+B'}{2}\right). \qquad (26)$$

We can, similarly, replace the diagonal matrices $D(\lambda)$, $D(\delta)$, $D(\gamma)$ and $D(\mu)$, respectively, by $\lambda_{min}I$, $\delta_{max}I$, $\gamma_{min}I$ and $\mu_{max}I$ so that the right-hand side of (22) decreases (at least it does not increase), and can prove that

$$\frac{c(AB) - \bar{c}(AB)}{2i} \geq$$

$$c_{min}\left(\frac{A+A'}{2}\right)c_{max}\left(\frac{B-B'}{2i}\right) + c_{min}\left(\frac{A-A'}{2i}\right)c_{max}\left(\frac{B+B'}{2}\right). \qquad (27)$$

Combining (26) and (27), we have (25).

<u>Theorem</u> 6. <u>Let A and B be two commuting n-square real matrices, such that</u> $(A+A')/2$ <u>is indefinite and</u> $(B+B')/2$ <u>is at least positive semi-definite. Then</u>

(a). <u>if</u> $c_{max}\left(\frac{A+A'}{2}\right) < -c_{min}\left(\frac{A+A'}{2}\right)$,

$$c_{min}\left(\frac{A+A'}{2}\right)c_{max}\left(\frac{B-B'}{2i}\right) + c_{min}\left(\frac{A-A'}{2i}\right)c_{max}\left(\frac{B+B'}{2}\right)$$

$$\leq \frac{c(AB) - \bar{c}(AB)}{2i}$$

$$\leq c_{min}\left(\frac{A+A'}{2}\right)c_{min}\left(\frac{B-B'}{2i}\right) + c_{max}\left(\frac{A-A'}{2i}\right)c_{max}\left(\frac{B+B'}{2}\right),$$

(b). <u>otherwise</u> (i.e., if $c_{max}\left(\frac{A+A'}{2}\right) \geq -c_{min}\left(\frac{A+A'}{2}\right)$)

$$c_{max}\left(\frac{A+A'}{2}\right)c_{min}\left(\frac{B-B'}{2i}\right) + c_{min}\left(\frac{A-A'}{2i}\right)c_{max}\left(\frac{B+B'}{2}\right)$$

$$\leq \frac{c(AB) - \bar{c}(AB)}{2i} \tag{29}$$

$$\leq c_{max}\left(\frac{A+A'}{2}\right)c_{max}\left(\frac{B-B'}{2i}\right) + c_{max}\left(\frac{A-A'}{2i}\right)c_{max}\left(\frac{B+B'}{2}\right),$$

<u>where c_{max} and c_{min} stand, respectively, for the greatest and least characteristic roots.</u>

The inequalities (28) and (29) can be obtained from (22) by replacing the diagonal matrices $D(\lambda)$, $D(\vartheta)$, $D(\gamma)$ and $D(\mu)$ by suitable scalar matrices and by using arguments parallel to those used in establishing the inequalities (21) and (25). But, while proving, this is to be remembered that for the indefinite matrix $(A+A')/2$, $c_{max}\left(\frac{A+A'}{2}\right) > 0$, $c_{min}\left(\frac{A+A'}{2}\right) < 0$, and if $c_{max}\left(\frac{A+A'}{2}\right) < -c_{min}\left(\frac{A+A'}{2}\right)$, $c_{max}\left(\frac{A+A'}{2}\right)c_{max}\left(\frac{B-B'}{2i}\right) < c_{min}\left(\frac{A+A'}{2}\right)c_{min}\left(\frac{B-B'}{2i}\right)$ since $c_{max}\left(\frac{B-B'}{2i}\right) = -c_{min}\left(\frac{B-B'}{2i}\right)$. Due to this fact the case, when $c_{max}\left(\frac{A+A'}{2}\right) < -c_{min}\left(\frac{A+A'}{2}\right)$, has been considered separately.

Similar results, giving lower and upper limits to the real and imaginary parts of $c(AB)$, can be established when $(B+B')/2$ is at least n.s.d. or indefinite and $(A+A')/2$ is at least p.s.d., n.s.d., or indefinite.

In order to see that the condition, $AB = BA$, imposed on the matrices in the results established in this paper, is necessary, we consider the following one example:

Let $A = \begin{pmatrix} 0 & 1 \\ 2 & 0 \end{pmatrix}$ $B = \begin{pmatrix} 2 & 2 \\ 1 & 2 \end{pmatrix}$, so that

$$AB = \begin{pmatrix} 1 & 2 \\ 4 & 4 \end{pmatrix} \neq BA.$$

Here $(A+A')/2$ is indefinite with characteristic roots 3 and -3; and $(B+B')/2$ is positive definite with characteristic roots $1/2$ and $7/2$. Also $c(\frac{A-A'}{2i}) = \pm\frac{1}{2}$ $= c(\frac{B-B'}{2i})$ and $c(AB) = (5 \pm \sqrt{41})/2$. Since $(5+\sqrt{41})/2$ is not less than or equal to $11/2$, Theorem 3 is not true for these non-commutative matrices A and B.

4. Some particular cases of the above theorems.

We now consider certain examples to show that some interesting results can be obtained from the above theorems as particular cases.

(i). If we put $B = I$ for which $(B+B')/2 = I$ and $B-B' = 0$, we see that (1), (15) and (18) reduce to

$$c_{min}\left(\frac{A+A'}{2}\right) \leq \frac{c(A) + \bar{c}(A)}{2} \leq c_{max}\left(\frac{A+A'}{2}\right),$$

a result due to Bendixon [1], giving the lower and upper limits for the real part of c(A) in terms of the characteristic roots of $(A+A')/2$, when $(A+A')/2$ is at least p.s.d., n.s.d. or indefinite.

(ii). If we put $B = I$ in (21), (25), (28) and (29), we have the following well-known result

$$c_{min}\left(\frac{A-A'}{2i}\right) \leq \frac{c(A) - \bar{c}(A)}{2i} \leq c_{max}\left(\frac{A-A'}{2i}\right),$$

or $$\left| \frac{c(A) - \bar{c}(A)}{2i} \right| \leq \max \left| c\left(\frac{A-A'}{2i}\right) \right| ,$$

due to Bromwich [2] .

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